MAXIMAL SETS

(A survey on the maximal sets inside
the theory of the recursively enumerable
sets and their degrees)

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PREFACE

In the year 1958 Friedberg showed that there are maximal sets and answered with this a question of Myhill from the year 1956. This result and at this above all the method of the construction were very important for the theory of recursively enumerable sets. Since this time the maximal sets, their properties and the construction principle of these sets as well as generalizations of the maximal sets were investigated in many papers .

The maximal sets, hence those recursively enumerable sets with the most eas~~y~~iest structure of the recursively enumerable supersets were, very oftenly for many different notions of the recursion theory the first recursively enumerable sets to which the relationship with these notions were analysed. Thus the maximal sets belong to those recursively enumerable sets which are at most investigated .

In this paper will be given a representation of all known (by the author) results on maximal sets in a systematical order, but without proofs, together with new results. It will be done similar as the summaries /So,78/ or /Od/ .

A first survey about the maximal sets and their properties inside all recursively enumerable sets was given by Rogers in /Ro/ Chapter XII in the year 1965. But in the last 20 years there was proved many further states of affairs. Thus that what is is /Ro/ about the maximal sets is far from current knownledge.

We present here not only results about the maximal sets but treat here also related topics.

We are doing this for two reasons. First we think that this is useful for seeing which place the theory of ~~sets~~ maximal sets has inside the theory of all recursively enumerable sets. Secondly we want to present topics of the theory of all recursively enumerable sets, but restricted to a subfamily and thus still representable by a paper. The theory of the maximal sets is of course an essential restriction of the theory of all recursively enumerable sets, since many notions became are interesting only for larger classes of recursively enumerable sets. But the theory of all these sets is already now to voluminous even when it is presented only in the form of a survey.

As we shall see what is already known about maximal sets is quite extensive.

Since maximal sets have many properties in common and since they have an easy to characterize lattice structure, an interesting point of view inside the analysis of these recursively enumerable sets is to find new recursion theoretic notions and investigate the relationship between these and the maximal sets.

Besides the results the paper also includes questions and propblems by which the maximal sets can be analysed further.

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INTRODUCTION

For the representation of the results in this paper a lot of notions and symbols are necessary. The most fundamental and in many places used are given here in the beginning of the paper.

1. Set - theoretic symbols

We use the usual set - theoretic notions $\cap$, $\cup$ and $\setminus$ for the intersection , union and difference of two sets respectively and $\subseteq$ for the inclusion between two sets. If x is an element of the set X we write $x \in X$.

If X and Y are sets then with $X \times Y$ we denote the set of pairs of elements of X and Y ($X \times Y = \{(x,y) : x \in X \wedge y \in Y\}$) . In stead of $(X \times (X \times \ldots (X \times X)) \ldots))$ - n times the product of X we write X^n and for the elements of X^n $(x_1,\ldots,x_n)$.

ω is used as the symbol for the set of natural numbers i.e. $\{0,1,2,\ldots\}$ and $\mathcal{P}(\omega)$ for the power set of ω (in general $\mathcal{P}(X)$ - the power set of the set X).
$\emptyset$ means the empty set . $\overline{X}$ is only used for subsets of ω and denotes the complement of X .

Let X and Y be two subsets of ω . With $X \oplus Y$ we denote the set

$$\{2n : n \in X\} \cup \{2n+1 : n \in Y\} \quad .$$

Let m and n be two numbers with $m \leq n$. With $[m,n]$, $[m,n)$, $(m,n]$, (m,n) we denote the sets

$$\{l : m \leq l \leq n\} , \quad \{l : m \leq l < n\} , \quad \{l : m < l \leq n\}$$
$$\{l : m < l < n\}$$

respectively.
A subset X of ω is called initial set if $X = [0,n]$ for some n or if $X = \omega$.

If X is a set then with card(X) or also with $|X|$ we denote the cardinality of X (i.e. $|X| = 0,1,2,\dots$, $|X| < \infty$ means that X is finite , $|X| = \infty$ that X is infinite , $|X| = \omega$ that X is countable and $|X| = 2^{\aleph_0}$ that X has the cardinality of Continuum) .

With $\max|X|$, for finite sets X , we denote the greatest element of X ($\max|\emptyset| \underset{df}{=} 0$) .

If $X \subseteq Y$ with $Y \setminus X$ infinite we write shortly $X \subset_{\infty} Y$. If P is any property defined in $\mathfrak{P}(\omega)$ we say that a set is co-P- if the complement of this set has the property P (e.g. cofinite , coinfinite and others) .

Fin means the class of finite subsets of ω and Cof the class $\{X \in \mathfrak{P}(\omega) : \overset{\frown}{X} \in \mathrm{Fin}\}$.

If $\mathfrak{X}$ is a subclass of $\mathfrak{P}(\omega)$ then $\mathfrak{X}^{0}$ means the class $\mathfrak{X} \cup \mathrm{Cof}$.

The equality between two sets modulo finite differences plays an important role in the paper. We write for two sets $X , Y \in \mathfrak{P}(\omega)$ $X =^{*} Y$ or $X=Y(\mathrm{mod\,Fin})$ if $(X \setminus Y) \cup (Y \setminus X)$ is a finite set .
$X \subset^{*} Y$ means that $X \subseteq Y \cup Z$ for some finite set Z or equivalent that $X \setminus Y$ is finite .
The relation $=^{*}$ is an equivalence relation in every subclass $\mathfrak{X}$ of $\mathfrak{P}(\omega)$. Denote with X^{*} the equivalence class resp. to $=^{*}$ including X and with $\mathfrak{X}^{*}$ the class $\{X^{*} : X \in \mathfrak{X}\}$.
Observe that for $\mathfrak{X} \subseteq \mathfrak{P}(\omega)$, $\mathfrak{X}^{*} \subseteq \mathfrak{P}^{*}(\omega)$ means that every equivalence class of $\mathfrak{X}^{*}$ is closed under $=^{*}$, i.e.
$X \in \mathfrak{X} \wedge X =^{*} Y \longrightarrow Y \in \mathfrak{X}$.
(Instead of $\cap^{*}$ and $\cup^{*}$ we write only $\cap$ and $\cup$ resp.) .

Let f be a partial function from X^{n} into a set Y . dom(f) denotes the set of tuples from X^{n} for which f is defined and rg(f) the set of elements of Y which are ranges of tuples from dom(f) by f . $f | X$ means the restriction of the function f to the set X.

Let R be a relation in ω^{k} . With C_{R} we denote the characteristic function of R , i.e.

$$C_R(x_1,\dots,x_n) = \begin{cases} 0 : R(x_1,\dots,x_n) \text{ holds} \\ 1 : \text{otherwise} \end{cases} .$$

(We identify subsets with the unary relations corresponding to these sets and write C_X for $X \in \mathfrak{P}(\omega)$.

If φ is a partial function and x a number , then we write $\varphi(x)\downarrow$ if φ is defined for x .
Let φ_1 and φ_2 be two partial functions. The relation between both is defined as follows : $\varphi_1(x) \simeq \varphi_2(x)$ means that either both are defined and then $\varphi_1(x) = \varphi_2(x)$ or both are not defined .

If f and g are two functions , then $f \oplus g$ is the function

$$(f\oplus g)(n) = \begin{cases} g(k) & : 2k = n \\ f(k) & : 2k+1 = n \end{cases} .$$

2. Recursion theory

We shall use in the paper also a lot of recursion theoretic notions , symbols and results . For this reason the mostly used notations together with few basic results from the recusion theory will be given here .

2.1 (Basic notions) The fundamental class of sets , which is regarded in the paper is the class of recursively enumerable sets . For recursively enumerable we write shortly r.e. . For the lattice of r.e. sets we use the symbol $\mathcal{E}$.

If $\mathcal{L}$ is a distributive lattice with 0 and 1 elements , then with $\mathcal{L}_r$ we denote the sublattice of $\mathcal{L}$ consisting of all elements having a complement in $\mathcal{L}$.
Well-known is that $\mathcal{E}_r$ consists exactly of all recursive sets.

Let X and Y be r.e. sets with $X \subseteq Y$. With $\mathcal{E}(X,Y)$ we denote the sublattice of $\mathcal{E}$ consisting of all r.e. sets laying between X and Y . For $\mathcal{E}(X,\omega)$ we write shortly $\mathcal{E}(X)$ and for $\mathcal{E}(\emptyset,X)$ the symbol $\mathcal{E}\restriction X$.
By meaning of $\mathcal{E}(X,Y)$ and the relation $=^*$ we can define

the following factor structures :

$$\mathcal{E}^*,\ \mathcal{E}^*_r,\ \mathcal{E}^*(X),\ \mathcal{E}^*_r(X),\ (\mathcal{E}|X)^*,\ (\mathcal{E}|X)^*_r,$$
$$\mathcal{E}^*(X,Y),\ \mathcal{E}^*_r(X,Y) .$$

Behind cofinite and coinfinite the notion co-r.e. also is oftenly used . By the arrangement a set X is co-r.e. if $\overline{X}$ is a r.e. set .

We write $X \equiv Y$ for two sets X and Y if there is a recursive permutation of ω with $p(X) = Y$ and $X \equiv^* Y$ if $X \equiv Z$ and $Z =^* Y$ for some set Z .

2.2 (Pairing function) If $(x_o,\ldots,x_{n-1})$ is a sequence of numbers , then with $\langle x_o,\ldots,x_{n-1}\rangle$ we denote the sequence number of this sequence and converse for numbers a and i we write $(a)_i$ for the i-th element of the sequence connected with a (e.g. $(\langle n,m\rangle)_o = n$, $(\langle n,m\rangle)_1 = m$) .
Not every number is a sequence number , but we claim that $(a)_i$ is defined for all a and i .

2.3 (Recursive arrays) A sequence of r.e. sets $(U_n)_{n\geq o}$ is called <u>recursive array</u> if the (binary) relation $\{(x,n) : x \in U_n , n \geq o\}$ is r.e. .
We say that a recursive array $(U_n)_{n\geq o}$ satisfies the S_{mn}- Theorem if $(U_n)_{n\geq o}$ is " universal resp. to recursive embeddings " , i.e. if for every recursive array $(V_n)_{n\geq o}$ there is a recursive function f , s.t. $V_n = U_{f(n)}$ for all $n \geq o$.
Recursive arrays satisfying the S_{mn}- Theorem are called <u>acceptable arrays</u> or also <u>standard arrays</u> . For such arrays important properties such as the Recursion Theorem are satisfied.

With $(W_e)_{e\geq o}$ we denote an acceptable array . (Since all acceptable arrays are recursively invariant, the choice of $(W_e)_{e\geq o}$ is not important) .

A recursive array $(U_n)_{n\geq o}$ is called disjoint if all sets U_n , U_m $(n\neq m)$ are pairwise disjoint. Further we say that $(U_n)_{n\geq o}$ is a finite array if all U_n are finite .

A sequence of partial recursive functions $(\rho_i)_{i \geq 0}$ is called a recursive f-array if the function ρ with $\rho(x,y) \simeq \rho_x(y)$ is partial recursive .
Analogously as for r.e. sets there are also recursive f-arrays which are universal resp. to recursive embeddings. Denote with $(\varphi_e)_{e \geq 0}$ an acceptable f-array. Then $(\varphi_e)_{e \geq 0}$ also satisfies the Recursion Theorem , see e.g. /Ro,Chapter XI/ . We can claim that $W_e = \mathrm{dom}(\varphi_e)$ for all $e \geq 0$.

Simultaneous enumerations of recursive arrays

Let X be an infinite r.e. set. A function g is called an enumeration of X if g is recursive, injective and rg(g) =X. We say that the sequence $x_0, x_1, \ldots$ is an enumeration of X if the function g with $g(n) = x_n$ is an enumeration of X . (If X is finite , then an enumeration of X is a(n) (arbitrary) finite sequence $x_0, x_1, \ldots, x_{n-1}$ of all elements of X without repetition.
Let $(U_n)_{n \geq 0}$ be a recursive array . Then there are recursive functions g , s.t. for every pair (x,y) with $x \in U_y$, $y \geq 0$ the number $\langle x,y \rangle$ appears exactly one time as $g(z)$ for some z .

Such functions are called simultaneous enumerations of $(U_n)_{n \geq 0}$.

Let $(U_n)_{n \geq 0}$ be a recursive array and g a simultaneous enumeration of $(U_n)_{n \geq 0}$. With $U_{n,s}$ we denote the set $\{x : (\exists t < s)((g(t))_0 = x \wedge (g(t))_1 = n\}$.
In the most constructions of r.e. sets the used simultaneous enumeration of $(U_n)_{n \geq 0}$ (in particulary for the array $(W_e)_{e \geq 0}$) is not important. But for the automorphism proof in point 3 the properties of the used recursive arrays and the simultaneous enumerations of these arrays are essentional .

Let $X_0, X_1, \ldots, X_{l-1}$ be r.e. sets and $(U^0_n)_{n \geq 0}$, $(U^1_n)_{n \geq 0}$, ..., $(U^{k-1}_n)_{n \geq 0}$ be recursive arrays .
We call a function g simultaneous enumeration of these sets and arrays if g is injective ,

$$(\forall x)(\forall y)(\ x \in X_y \longleftrightarrow \langle x,y,0 \rangle \in rg(g)\ )$$

$$(\forall x)(\forall y)(\forall z)(\ x \in U_y^{z} \longrightarrow \langle x,y,z+1\rangle \in rg(g)\ )$$

holds and every element of rg(g) has the form $\langle x,y,i\rangle$, $(i=0 \wedge x \in X_y) \vee (i \geq 1 \wedge x \in U_y^{i+1})$.

Let g be a simultaneous enumeration of r.e. sets and recursive arrays . Suppose $(U_n^i)_{n \geq 0}$ is one of these arrays . We indroduce the following notations :

$$U_n^i -_g U_m^i = \{ x : \langle x,n,i+1\rangle \text{ is earlier enumerated by } g \text{ as } \langle x,m,i+1\rangle \}$$

$$U_n^i \searrow_g U_m^i = (\ U_n^i -_g U_m^i\ ) \cap U_m^i \quad .$$

2.4 (Simple sets , Δ_2^0 sets) One fundamental subclass of r.e. sets is the family of the simple sets. We use the symbol $\mathcal{S}$ for the class of simple sets.

A simple set S is called Post's simple set if

$$(\forall e)(\ (\exists x)(\ x \in W_e \wedge x > 2e\ ) \longrightarrow W_e \cap S \neq \emptyset\ ) \quad .$$

Simple sets with this property have special properties which are used in some places in the following .

Behind the partial recursive functions and the r.e. sets for the presentation of the results we need still more general objects , namely the arithmetical functions and sets.

One class of arithmetical sets and a way of representing these is very oftenly used in the proofs of the Theorems in this paper.

The next more general class of the class of r.e. sets is the class of Δ_2^0 sets .

A subset X of ω is Δ_2^0 if there are recursive relations R and Q , s.t.

$$n \in X \longleftrightarrow (\exists x)(\forall y)R(n,x,y)$$
$$n \notin X \longleftrightarrow (\exists x)(\forall y)Q(n,x,y) \quad .$$

The following equivalence to the definition is well-known :
$X \in \Delta_2^0$ iff there is a 2-ary function g , recursive with $rg(g) \subseteq \{0,1\}$, s.t.

$$n \in X \longleftrightarrow (\exists s_0)(\forall s > s_0)(\ g(x,s) = 0\ )$$
$$n \notin X \longleftrightarrow (\exists s_0)(\forall s > s_0)(\ g(x,s) = 1\ ) \quad .$$

3. Finite 01-sequences

A fundamental tool for the whole recursion theory are the finite sequences consisting of 0 and 1 together with their properties. These objects and the relations between these are very obviously and those they are oftenly used in constructions , e.g. also of r.e. sets .

Instead of finite 01-sequences we also say , 01-words, strings or states (depending of the manner of use) .

The set of all finite 01-sequences is denoted with $2^{<\omega}$ (i.e. $2^{<\omega} = \{(a_0,a_1,\dots,a_{n-1}) : a_i \in \{0,1\}\ ,\ n \geqslant 1\} \cup \{\emptyset\}$, where $\emptyset$ is the symbol for the empty sequence) .
As variables on the set $2^{<\omega}$ we use small greek letters : $\sigma, \tau, \vartheta, \dots, \sigma_1, \sigma_2, \dots$.

Let $\sigma \in 2^{<\omega}$. Hence $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_{k-1})$ for some $k \geqslant 1$ or $\sigma = \emptyset$. With $|\sigma|$ we denote the length of σ . Thus $|\sigma| = k$ ($|\emptyset| = 0$) and $\sigma(i) = \sigma_i$ for $i=0,1,\dots,k-1$. If $l \leqslant |\sigma|$, then $\sigma | l = (\sigma_0,\dots,\sigma_{l-1})$ ($\sigma | 0 = \emptyset$) .

The combination of two members of $2^{<\omega}$ we denote with $*$ (i.e. if $\sigma = (\sigma_0,\dots,\sigma_{k-1})$, $\tau = (\tau_0,\dots,\tau_{l-1})$ then $\sigma * \tau$ is $(\sigma_0,\dots,\sigma_{k-1},\tau_0,\dots,\tau_{l-1})$). In particular we have the combinations $\sigma * 0$ and $\sigma * 1$ for every $\sigma \in 2^{<\omega}$.

$\sigma \preccurlyeq \tau$ for $\sigma, \tau \in 2^{<\omega}$ means that there is a $\vartheta \in 2^{<\omega}$ with $\sigma * \vartheta = \tau$ (σ is an <u>initial part</u> of τ or equivalent $\sigma = \tau | |\sigma|$).
Denote with $\mathcal{T}$ the system $(2^{<\omega}, \preccurlyeq)$. Let $\sigma \in 2^{<\omega}$. $\mathcal{T}[\sigma]$ means the set $\{\mathcal{T} \in 2^{<\omega} : \sigma \preccurlyeq \tau\}$.

Lexicographically order of $2^{<\omega}$

We define a linear order in $2^{<\omega}$ which is called <u>lexicographical order</u> (of $2^{<\omega}$) and is denoted with $<^*$. Let

$\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{k-1})$ and $\tau = (\tau_0, \tau_1, \ldots, \tau_{l-1})$ be elements of $2^{<\omega}$. $\sigma <^* \tau$ holds if

$$(k<l) \vee (k = l \wedge (\exists i<k)(\sigma|i = \tau|i \wedge \sigma(i)=0 \wedge \tau(i)=1)).$$

Thus $\emptyset <^* 0 <^* 1 <^* 00 <^* 01 <^* 10 <^* \ldots$.

By using $<^*$ we can define a bijection between $2^{<\omega}$ and ω by $\sigma \longrightarrow \mathrm{card}(\{\tau : \tau <^* \sigma\})$.

Let $\hat{\sigma}$ be the range of σ and st^n be the element of $2^{<\omega}$ having n as range .

This bijective mapping allows us also to speak about recursive functions from $2^{<\omega}$ in $2^{<\omega}$ or from $2^{<\omega}$ in ω and about r.e. sets in $2^{<\omega}$, as also about Σ^0_n sets in $2^{<\omega}$, for $n=2,3,\ldots$ when we replace $\sigma \in 2^{<\omega}$ by $\hat{\sigma} \in \omega$.

We use the finite 01-sequences , their properties and the relations between these in the following different ways :

i) (Coding sets of numbers). A subset Ω of $2^{<\omega}$ is called <u>branch</u> if $\Omega \neq \emptyset$ and

$$\sigma \in \Omega \wedge \tau \preceq \sigma \;.\longrightarrow \tau \in \Omega$$
$$\sigma \in \Omega \longrightarrow ((\sigma * 0 \in \Omega \vee \sigma * 1 \in \Omega) \wedge \neg(\sigma * 0 \in \Omega \wedge \sigma * 1 \in \Omega)) .$$

Every subset X of ω corresponds in a unique way to a branch of $2^{<\omega}$.

Let $X \in \mathcal{P}(\omega)$. Then $\sigma(X)$ means the branch

$$\{\sigma \in 2^{<\omega} : \sigma = \emptyset \vee (\forall i < |\sigma|)(\sigma(i) = 1 \longleftrightarrow i \in X)\}.$$

With $\sigma(X,n)$ for $X \in \mathcal{P}(\omega)$ and $n \in \omega$ we denote the (unique) string τ with $\tau \in \sigma(X)$ and $|\tau|=n$.

ii) (Enumeration measure) . Suppose there is given a recursive array $(U_n)_{n \geq 0}$ together with a simultaneous enumeration g of $(U_n)_{n \geq 0}$. Then it can be defined a <u>state-function</u> st with three variables by

$$st(x,n,s) = \sigma \text{ with } |\sigma| = n \text{ and for } i < n \quad \sigma(i) = 1 \longleftrightarrow x \in U_{i,s}$$

(where $U_{i,s}$ is enumerated by g).

We see that

$$st(x,n,s) <^* st(x,n,s+1) \quad .$$

Denote with $st(x,n)$ the final state, i.e. $\lim_s st(x,n,s)$. Hence

$$st(x,n) = \sigma \text{ iff } |\sigma| = n \wedge (\forall i < n)(\sigma(i) = 1 \longleftrightarrow x \in U_i).$$

Let $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ from $2^{<\omega}$. With $\bigwedge W(\sigma)$ we denote the set

$$W_0^{\sigma(0)} \cap \ldots \cap W_{n-1}^{\sigma(n-1)} \qquad (W_i^0 = W_i \; , \; W_i^1 = \overline{W_i}) \; .$$

This abbreviation will be used later in few places .

iii) (Arrangement of numbers). There are constructions of r.e. sets in which in the beginning the numbers are placed on the elements of $2^{<\omega}$. (To every element of $2^{<\omega}$ or of a subset of $2^{<\omega}$ exactly one number) and while the construction the numbers are moved from one element to the second one. In other words, in the beginning a function from $2^{<\omega}$ onto ω is defined (such functions are called number trees) and in every step of the construction a new number tree is defined by using the preceding one respectively to special rules.

This is oftenly a very usefull method to ensure that the constructed r.e. set has the desired properties. (For one application of this see e.g. /La,68a/. In this case we say strings for the finite 01-sequences.

iv) (Objects for themselves). In the first three cases the finite 01-sequences were connected with the numbers. But these objects for themself together with $\preccurlyeq$ are also usefull.

Take $\mathfrak{T}$ (equal to $(2^{<\omega}, \leq)$). We say that $\Omega \subseteq 2^{<\omega}$ is an *ideal* (in $\mathfrak{T}$) if

$$\sigma \in \Delta \wedge \sigma \leq \tau . \rightarrow \tau \in \Delta$$
$$\sigma * 0 \in \Delta \wedge \sigma * 1 \in \Delta . \rightarrow \sigma \in \Delta \quad .$$

(So e.g. $\Delta = \emptyset$, $\Delta = \mathfrak{T}[\sigma]$, $\Delta = 2^{<\omega}$, ... are ideals in $\mathfrak{T}$).

If Δ_1 and Δ_2 are ideals then of course also $\Delta_1 \cap \Delta_2$ is it. Denote with $\Delta_1 \oplus \Delta_2$ the smallest ideal including both. Then the class of all ideals together with $\cap$ and $\oplus$ forms a distributive lattice.

We say that a subset Δ of $2^{<\omega}$ belongs to the class Σ^0_n if $\{ \tilde{\sigma} : \sigma \in \Delta \}$ is an Σ^0_n set .

Denote with $\mathcal{E}^n_I$ the class of Σ^0_n ideals with the operations $\oplus$ and $\cap$.

$\mathcal{E}^n_I$ is a distributive lattice for n=1,2,... (Observe that $\Delta_1 \oplus \Delta_2$ also is Σ^0_n if Δ_1 and Δ_2 are Σ^0_n) .

We need later in particulary $\mathcal{E}^1_I$ for which we write $\mathcal{E}_I$.

4. Logic

In addition to the usual logical symbols the following formulations and symbols are still used :

We write " for almost all n "...P... if all numbers except a finite number of them have this property ...P... and use the abbreviation

$$(\forall \text{a.a.n})P$$

The notation

$$(\exists^{\infty} n)P$$

means that infinitely many numbers have the property P.

Further we write $(\forall X \text{r.e.})$ (for all r.e. sets) , $(\exists R \text{rec.})$ (there is a recursive set) , $(\forall f \text{ par.rec.fcts})$ (for all partial recursive functions) and so on .

1. THE BASIC CONSTRUCTION OF MAXIMAL SETS

We start our investigations on the maximal sets with the basic construction of maximal sets done by Friedberg. Since this construction principle is used in all other special constructions of maximal sets, it will be given a sketch of the proof. Further we analyse this construction under different points of view and give lattice theoretic consequences which follows directly from the existence of maximal sets.
In the second subpoint we desribe modifications of this construction principle. These modifications are used in Theorems in which special maximal sets are constructed.

1.1 Definition and existence of maximal sets .

Definition 1.1.1: A subset X of ω is called maximal if X is r.e. coinfinite and

$$(\forall e)(X \subseteq W_e \longrightarrow (X =^* W_e \vee W_e =^* \omega)).$$

This definition and the question for the existence came from Myhill /My/ 1956. Friedberg showed 1958 in /Fr/ that there are such special r.e. sets.

Theorem 1.1.2(Friedberg): There are maximal sets

(Sketch of the proof). The existence of maximal sets will be shown by giving a construction procedure for such r.e. sets . The construction will be provided stepwise. With M_s we denote the set which after s steps is already constructed (M_o is put equal to $\emptyset$). In every step s it is known whether new elements come to M_s and which numbers these are or if not. From this it follows that the set $M = \bigcup_{s \geq 0} M_s$ is r.e. . From the construction instruction given in the following it can be proved that then M is maximal.

In addition to the set M_s it will be still constructed in every step a function m^s. m^s will be a strongly monoton increasing function with $rg(m^s) = \overline{M_s}$ for every $s \geq 0$.

step 0. Put $M_0 = \emptyset$ and $m^0(i) = i$, $i \geq 0$.

step s+1: We assume that M_s and m^s are already constructed with the properties mentioned above.
Look if there are numbers k , i and j such that

$$(1.1) \qquad (k \leq i < j) \wedge (st(m^s(i),k,s) <^* st(m^s(j),k,s)) \quad .$$

We see that the relation (1.1) is recursive (in k,i,j,s) , since for alls the numbers x and y for which st(x,y,s) consists not only of 0 are known and these are finitely many.

If such numbers k , i and j do not exist let $M_{s+1} = M_s$ and $m^{s+1}(i) = m^s(i)$ for all i .
But if there are triples of numbers satisfying (1.1) take that triple among all those with the smallest i , for this i the smallest k and then the smallest j . Let (i_0, j_0, k_0) be this triple.
Now we define

$$\begin{aligned} M_{s+1} &= M_s \cup \{m^s(i_0)\} \\ m^{s+1}(e) &= m^s(e) && : \quad e < i_0 \\ &= m^s(e+(j_0-i_0)) && : \quad i_0 \leq e \end{aligned} \quad .$$

The set M ($= \bigcup_{s \geq 0} M_s$) constructed in this way is maximal.
The proof of this is divided into two parts.

First it can be shown by induction that $\lim_s m^s(i)$ exists for all i . At this it is essential that in (1.1) we have $<^*$(and not $\leq^*$) and also that $k \leq i < j$. Since for all sufficiently large steps s+1 $m^{s+1}(i) \neq m^s(i)$ implies $st(m^s(i),i,s) <^* st(m^{s+1}(i),i,s+1)$, $\lim_s m^s(i)$ must exist . Let $m(i)$ be the Limit number of $m^s(i)$, $s \geq o$. It is obvious that the function m is strongly increasing and that $\overline{M} = rg(m)$.

Next it can be shown again by induction that for every e either almost all of the numbers $m(o)$, $m(1)$,... belongs to W_e or only finitely many of them are elements of W_e .
We assume that this holds for all indizes $0,1,...,e-1$. Thus for all $x > x_o$ $m(x)$ have the same e-state for some x_o.
If there are numbers x_1,x_2 with $x_o \leq x_1 < x_2$ and $m(x_1) \notin W_e$, but $m(x_2) \in W_e$ then the triple (x_1,e,x_2) satisfies infinitely often (1.1) . Thus for some s (x_1,e,x_2) must be (i_o,k,j_o). This condradicts the assumption $m(x_1) \notin W_e$.
Hence for all e

$$\overline{M} \subset^* W_e \vee M \cap W_e =^* \emptyset \quad .$$

But this means that M is maximal. ■

For the class of all maximal sets we use the symbol <u>Max</u> .

The existence of maximal sets has special lattice theoretic consequences already mentioned in /My/.
A lattice is called <u>dense</u> if it holds

$$(\forall a)(\forall b)(a < b \longrightarrow (\exists c)(a < c < b)) \quad .$$

It is easy to see that $\mathcal{E}^*$ is dense iff $\mathcal{E}$ has no maximal sets /Ro/,p.294 . Thus from Theorem 1.12 it follows that $\mathcal{E}^*$ is no dense lattice.

Denote with $\mathcal{A}(\mathcal{E}^*)$ the smallest Boolean algebra which is generated from $\mathcal{E}^*$(i.e. all Boolean combinations of elements of $\mathcal{E}^*$).
The existence of maximal sets shows that $\mathcal{A}(\mathcal{E}^*)$ is not atomless. From this in particular it follows that $\mathcal{A}(\mathcal{E})$ is a proper subclass of the class of Δ_2^0 sets.

Let X be an infinite subset of ω and $x_0, x_1, \dots$ the sequence of the elements of X in order of magnitude, i.e. $x_0 < x_1 < \dots$. The function which assigns to the number n the element x_n is called principal function of X and is denoted with $p_X(n)$.

Observe that for maximal sets the number

$$\operatorname{card}(\{ n : C_{W_e}(p_{\overline{M}}(n)) \neq C_{W_e}(p_{\overline{M}}(n+1)) \})$$

is finite for all $e \geq 0$ and for the set constructed in Theorem 1.1.2 this number even is bounded by 2^{e+1} .

Manners of generating maximal sets

Game theory. In /La,70/ there was given a game theoretic method for constructing recursive arrays. The purpose of this is to get an uniform way for proving elementary properties of the lattice $\mathcal{E}$.
Among others there also is given a game theoretic description of the construction of a maximal set.

Splinter sets. If f is a recursive function and n a number , then we call the set $\{a, f(a), f(f(a)), \dots\}$ Splinter set of f and a and write for this set Spl(f,a) . Since f is recursive, splinter sets are r.e. . It is easy to see that simple sets cannot be splinter sets. Thus in particulary the maximal sets are not splinter sets. An analyse of the r.e. sets having such a form is given e.g. in /Ul,60/.
If we take functions from Δ^0_2 then we get all r.e. sets ,see /Pl/.

Ranges of recursive functions. Every nonempty r.e. set is equal to the range of a recursive function. Every recursive function is finitely generated by the well-known rules (superposition, definition by recursion and minimalization , starting from some basic functions).
A problem could be to find a generation of a recursive function having the range equal to a maximal set. One way of finding such a function could be a fine analyse of the construction

in Theorem 1.1.2 .

Domain of a partial recursive function. Analogously to the problem above is to find a partial recursive function with the domain equal to a maximal set.
Connected with this is to describe a Turing machine such that the set of (starting) words by which the machine is halting is a maximal set.

Diophantic equations. An equation $p(y_1,\dots,y_n) = 0$ is called diophantic equation, if p is a polynom and all coefficients of p are integers.
A problem is to find a diophantic equation $p=0$, such that the set

$$\{n \in \omega : (\exists y_2)(\exists y_3)\dots(\exists y_k)(p(n,y_2,y_3,\dots,y_k) = 0)\}$$

is maximal (where $\exists y_i$ means " there is an integers y_i ").

Maximal sets and the complement covering r.e. sets

Let g be a simultaneous enumeration of $(W_e)_{e \geqslant 0}$ and M a maximal set constructed as in Theorem 1.1.2 by using g.
When it holds $\overline{M} \subset^* W_e$ for an index e ? We give here a sufficient condition for this.

Let σ_l be the string of the length l which is the greatest respectively to $<^*$ inside the set

$$\{\sigma : |\sigma| = l \wedge (\cap W(\sigma)) -_g M \text{ is infinite } \} .$$

Lemma 1.13: 1) The strings $\emptyset$, σ_0 , $\sigma_1, \dots$ form an infinite branch in $(2^{<\omega}, \preccurlyeq)$,
2) For every e holds

$$\overline{M} \subset^* W_e \text{ iff } \sigma_{e+1}(e) = 1 .$$

Criterion 1.1.4: Suppose $W_e \subseteq (\bigcap W(\sigma_e) -_g M)$. Then

(1.2) $W_e -_g M$ is infinite iff $W_e \setminus M$ is infinite .

Without the assumption about W_e the implication from left to right in (1.2) does not hold in general.

This criterion is used particulary in the proof of Theorem 3.2.6.

Cohesive sets

A subset X of ω is called cohesive if X is infinite and

$$(\forall e)(X \wedge W_e =^* \emptyset \vee X \subset^* W_e) \quad .$$

We see at once that a set is maximal iff it is r.e. and its complement is a cohesive set. But not every cohesive set is the complement of a maximal set. Thus the theory of cohesive sets is not identical with that of the maximal sets. Therefore the cohesive sets will be analysed in general in point 10.

1.2 Variations of the construction principle.

The construction of maximal sets in Theorem 1.1.2 can be modified in different ways. These modifications make it possible that the constructed r.e. set has in addition to the maximality still further properties.

We give in this point in general form few of these modifications which are use in later Theorems about maximal sets.

1.2.1 Let be given a recursive function F with $F(n,s) \leq F(n,s+1)$ and $\lim_s F(n,s) < \omega$ for all $n \geqslant 0$.

Suppose in addition to the construction steps in 1.1.2 we provide additionally $F(i)$ times ($F(i) = \lim_s F(i,s)$) the construction step

$$(1.3) \qquad \begin{aligned} M_{s+1} &= M_s \cup \{ m^s(i) \} \\ m^{s+1}(j) &= m^s(j) \quad : \quad j < i \\ &= m^s(j+1) \quad : \quad i \leq j \end{aligned}$$

for every $i \geqslant 0$.
Since $F(i,s)$ is recursive and $F(i) < \omega$ for all i , the set such constructed also is maximal.

Remark. In the construction of so-called hyperhypersimple sets (for the definition of these sets , see 3.4) in /La,68/ there are used even recursive functions for which $F(i) = \omega$ for some i . In this case the construction with steps (1.3) gives no maximal set. For functions F with $F(i) = \omega$ for some i (but for infinitely many i $F(i) < \omega$) the construction must be changed to get a maximal set .

1.2.2 Let F be a function as in 1.2.1 . The construction of Theorem 1.1.2 can also be changed in the following way :

We still require for the triple (i,k,j) besides (1.1) that $F(k,s) \leqslant m^s(i)$.
Since F is recursive and $F(i) < \omega$ for all i , nevertheless we get a maximal set.

Observe that if F is recursive with $F(i,s) \leqslant F(i,s+1)$ then by applying the construction method 1.2.2 we get

$$(\forall n)(\ F(n) < \omega\ ) \longrightarrow M \text{ is maximal}$$

$$(\exists n)(\ F(n) = \omega\ ) \longrightarrow M \text{ is recursive} \qquad .$$

This fact is used in 4.2.4 and also in 9.3 .

1.2.3 In a special construction of a maximal set in subpoint 4.3 the elements of $\overline{M}_s$ are not linear ordered (of order type ω) but m^s will be a partial function defined in ω^2 with

$$(\forall j)(\forall i_0)(\forall i_1)(\ i_0 < i_1 \wedge m^s(i_1,j)\downarrow . \longrightarrow m^s(i_0,j)\downarrow).$$

In this construction the numbers are moved (i.e. $m^{s+1} \neq m^s$) only if

$$st(m^s(i,j),j,s) <^* st(m^s(k,l),j,s)$$

for $j < l$ independent of i and k .
If there is a function h (not necessary recursive) s.t.

$$(\forall j)(\forall s)(\forall i)(\ m^s(i,j)\downarrow \longrightarrow\ i \leq h(j)\ )$$

then the constructed set will be became maximal.

1.2.4 A 3-ary function V is called priority function if

1) V is a recursive function
2) $V(x,y,z) \leq V(x,y,z+1)$ for all x,y,z
3) $V(x,y,z) \leq h(y)$ for all x,y,z (h is some bounding function) .

An example of a priority function is the function W defined by

$$W(x,y,z) \underset{df}{=} \sum \{ 2^{y-i} : x \in W_{i,s} ,\ i \leq y \} .$$

If V is a priority function then it can be constructed quite analogously as in Theorem 1.1.2 a r.e. set if instead of (1.1) we take

$$(\ k \leq i < j\ ) \wedge (\ V(m^s(i),k,s) < V(m^s(j),k,s)\ ) .$$

We call such a set V-maximal . It is easy to see that W-maximal is the same as maximal. This follows from the equivalence

$$st(x,m,s) <^{*} st(y,m,t) \longleftrightarrow W(x,m,s) < W(y,m,t) .$$

It can be shown that some , not maximal , r.e. sets are V-maximal respectively to special priority functions V. (E.g. r-maximal sets or major subsets in /La,68a/ , for the definitions of these sets see subpoints 2.3 and 3.4) .

2. GENERAL RECURSION THEORETIC PROPERTIES OF THE MAXIMAL SETS

In this point it will be introduced many different subclasses of simple sets and properties of different kind and given an introducing analyse of them. Above all we are interested in the connection of these classes and properties to the class of maximal sets.
The purpose of this is to give an imagination of general properties of the maximal sets and of their position inside the class of simple sets. In point 2.3 it is given an summarized classification of all the subclasses and properties in the preceding points investigated. Further some of the properties of the maximal sets given in this point are needed in the proofs of other Theorems about the maximal sets mentioned in later points.

<u>2.1 Recursive arrays of disjoint sets.</u> At first we give a connection between maximal sets and recursive arrays which oftenly is used in the proofs of Theorems on maximal sets.

Theorem 2.1.1(Martin)/Ro,p.304/: Let M be a maximal set and $(S_n)_{n\geqslant 0}$ a recursive array of finite and disjoint sets. Then

$$(2.1)\quad (\forall \text{a.a. } n)(\ |S_n \cap \overline{M}| \leqslant 1\ ) \quad .$$

If $(S_n)_{n\geqslant 0}$ includes also infinite sets, then either in exactly one S_n lays $\overline{M}$ (mod Fin) or all sets $\overline{M}\cap S_i$ are finite and (2.1) holds.

Let $(S_n)_{n\geqslant 0}$ be as in Lemma 2.1.1 . Then of course for the most n we have $S_n \cap A = \emptyset$. The set $\{n : S_n \cap \overline{M} \neq \emptyset\}$ can be characterized completely by using the following Lemma :

Lemma 2.1.2(Madan/Robinson)/Mad,Ro/: Let W be a r.e. set and $(S_n)_{n\geqslant 0}$ a recursive array of disjoint sets. Then either

1° There is a disjoint recursive array $(T_i)_{i\geqslant 0}$ with $\bigcup_{i\geqslant 0} T_i = \bigcup_{i\geqslant 0} S_i$ and $T_i \cap \overline{W} \neq \emptyset$ for all i or

2° There is a recursive function g , s.t.

$$(\forall x)(x \in S_n \cap \overline{W} \longrightarrow x \leqslant g(n)) \qquad \text{for a.a.n} \quad .$$

Remarks 2.1.3. The proof of Lemma 2.1.2 allows to conclude stronger assertions from which other facts follow.

1) If for infinitely many n the sets $S_n \cap \overline{W}$ are infinite , then case 1° holds always

2) If for almost alln $S_n \cap \overline{W}$ is finite , then also if 1° holds $T_n \cap \overline{W}$ is finite for almost all n.

In particular the second of the remarks is used oftenly in proofs.

Corollary 2.1.4: If M is maximal and $(S_n)_{n\geqslant 0}$ a recursive array of disjoint sets then $\{n : \overline{M} \cap S_n \neq \emptyset\}$ is finite or recursive isomorphic with $\overline{M}$ (mod Fin) .

Thus starting from a maximal set M by meaning of recursive arrays $(S_n)_{n\geqslant 0}$ of disjoint sets we cannot construct new sets of the form $\{n : \overline{M} \cap S_n \neq \emptyset\}$. -

Characterizes the property (2.1) exactly the maximal sets (inside the coinfinite r.e. sets) ? In the next point 2.2 we show that this is not so.
Let

$$\mathcal{A}_{\leq 1} = \{X \text{ r.e. } : X \text{ is coinfinite } \wedge \text{ for all disjoint recursive arrys } (S_n)_{n \geq 0}\ (\forall a.a.i)\ (|S_i \cap \overline{X}| \leq 1)\}$$

$$\mathcal{A}^{f}_{\leq 1} = \{X \text{ r.e. } : X \text{ is coinfinite } \wedge \text{ for all disjoint recursive arrays } (S_n)_{n \geq 0} \text{ with } \bigcup S_n \supseteq \overline{X}\ (\forall a.a.i)(|S_i \cap \overline{X}| \leq 1)\} .$$

By Lemma 2.1.1 we get <u>Max</u> $\subseteq \mathcal{A}_{\leq 1}$ and from the definition $\mathcal{A}_{\leq 1} \subseteq \mathcal{A}^{f}_{\leq 1}$.
The relationship of the classes $\mathcal{A}_{\leq 1}$ and $\mathcal{A}^{f}_{\leq 1}$ to other classes of simple sets is given in the characterization in 2.3.4 .

2.2 Subclasses of simple sets and their relations to the class of maximal sets .

We introduce in this point further subclasses of simple sets and show their relation to the maximal sets. Already from this we can see how various the simple sets are and that the maximal sets form only a small subfamily inside all simple sets.

2.2.1 Quasimaximal sets

Starting from the maximal sets it can be found easily a greater class of simple sets. Thus, e.g. if M is maximal then $M \oplus M$ is not maximal but simple. This set belongs to the following family :

<u>Definition 2.2.1.1</u>: A r.e. set A is called <u>quasimaximal</u> (short , q-maximal) if A is coinfinite and $\overline{A}$ is an union of finitely many cohesive sets (if $\overline{A}$ is a so-called <u>quasicohesive</u> (q-cohesive) set).

The definition says just that A is q-maximal iff $\mathcal{E}^*(A)$ is a finite lattice (with more than one element).

It is obvious that all maximal sets are q-maximal and that all q-maximal sets are simple.
Denote the class of all q-maximal sets with $\mathcal{QM}$. The class $\mathcal{QM}^o$ forms a lattice definable filter in $\mathcal{E}$ and thus we can build up the factor lattice $\mathcal{E}/_{\mathcal{QM}^o}$.

Remark. In every distributive lattice there is a smallest congruence relation such that the corresponding factor lattice is dense (or consists of one element). It can be shown that the lattice $\mathcal{E}/_{\mathcal{QM}^o}$ still is not dense, see for this /A1/ .

The following Theorem gives an insight into the structure of the lattice $\mathcal{E}$ which also has a consequence for the q-maximal sets :

Theorem 2.2.1.2(Lachlan):/La68a/: Let A and B be r.e. sets with $A \subseteq B$ and $B \setminus A$ not co-r.e. . Then there is a recursive array $(S_n)_{n \geq o}$ of finite and disjoint sets with $S_n \cap (B \setminus A)$ not empty for all n and $A \cup \bigcup \{S_n : n \geq o\} = B$.

Let A and B be r.e. sets with $A \subseteq B$. Then we say that A is maximal in B if $A \subset_\infty B$ and for every $C \in \mathcal{E}(A,B)$ holds $A =^* C$ or $C =^* B$.

Corollary 2.2.1.3: If A is maximal in B then $B \setminus A$ is co-r.e. .

Proof: Suppose not and let $(S_n)_{n \geq o}$ be as in Theorem 2.2.1.2 for A and B . Then $A \subset_\infty A \cup \bigcup \{S_{2n} : n \geq o\} \subset_\infty B$. Thus A is not maximal in B .

Thus if A is maximal in B there is a recursive set R with $A \cup R = B$.

Corollary 2.2.1.4: A coinfinite r.e. set A is q-maximal iff $\mathcal{E}^*(A)$ is a finite Boolean algebra.

From this Corollary follows that every q-maximal set is equal to an intersection of a finite number of maximal sets. The smallest number of maximal sets whose intersection gives this q-maximal set is called *the order of this q-maximal set* .

The same analyse as at the end of 2.1 also can be provided for q-maximal sets.

Thus for every q-maximal set A of order n and every recursive array $(S_n)_{n \geq o}$ of finite and disjoint sets there is a number k with $k \leq n$ such that

$$(\forall a.a.n)(|\bar{A} \cap S_n| \leq k) .$$

This result can be still improved .

" Let $1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k \leq n$ with $\bigcup_{i=1}^{K} \alpha_i \leq n$ be a sequence of numbers. Then there is a q-maximal set A of order n and a recursive array $(S_n)_{n \geq o}$ as above with $\bar{A} \subseteq \bigcup\{S_n : n \geq o\}$

$$\{n : |\bar{A} \cap S_n| = \alpha\} \text{ is infinite iff } \alpha = \alpha_i ,\ i=1,\ldots,k".$$

Denote with $\mathcal{QM}_{\not\equiv^*}$ the q-maximal sets which are intersections of pairwise not $\equiv^*$- equivalent maximal sets. (i.e. for which $\alpha_k = 1$). We see that the class $\mathcal{QM}_{\not\equiv^*}$ is a proper super-class of *Max* and is included in $\mathcal{A}_{\leq 1}$.

2.2.2 Dense simple sets

A function f (an infinite set X) is called *dominant* (*dense immune*) if for all total recursive functions g holds

$$(\forall a.a.n)(\ g(n) \leq f(n)\ )\quad (\ (\forall a.a.n)(g(n) \leq p_X(n))\ ).$$

Definition 2.2.2.1(Martin,Tennenbaum)/So,78/: A r.e. set A is called *dense simple* if A is coinfinite and $\bar{A}$ is dense immune.

A r.e. set A is called *strongly dense simple* if A is coinfinite and for every partial recursive function g holds

$$(\forall a.a.n)(\ g(n)\downarrow \longrightarrow (\ g(n) \leq p_{\bar{A}}(n)\ )\ )\ .$$

This intensification of the notion dense simple implies an intersesting property which is given later in point 4.5 .

Theorem 2.2.2.2 (Martin): Every maximal set is dense simple.

But is is not true that every maximal set also is strongly dense simple. There are such maximal sets but not all , see for this 4.5 .

Denote with $\mathcal{DS}$ the class of dense simple sets and $s\mathcal{DS}$ the class of strongly dense simple sets.
Obviously $s\mathcal{DS} \subseteq \mathcal{DS}$. Both classes $\mathcal{DS}^{o}$ and $s\mathcal{DS}^{o}$ are filters in $\mathcal{E}$, $\mathcal{DS}^{o}$ is not cofinal in $\mathcal{E}$, i.e. not every coinfinite r.e. set is included is a dense simple set and further $\mathcal{DS}^{o}$ is not lattice definable, see /St,82a/.

2.2.3 Uniformly dense simple sets

In /Dë;71/ was introduced a notion which is properly stronger than the notion of dense simple set.

Definition 2.2.3.1(Dëgtev): An infinite set X is called *uniformly dense immune* if for all (total) recursive functions f hold

$$(\exists\ n)(\forall m)(\ n < m \longrightarrow f(p_X(m) < p_X(m+1)\ )\ .$$

The corresponding intensification to the class of all partial recursive functions we call *strongly uniformly dense immune* . This is the condition

$$(\forall f \text{ par. rec. fct})(\exists\, n)(\forall m)(\ n < m \longrightarrow (f(p_X(m)\downarrow \longrightarrow (f(p_X(m) < p_X(m+1))))\ .$$

A set X is called *uniformly dense simple* if X is r.e. and $\overline{X}$ is uniformly dense immune. Analogously the notion of *strongly uniformly dense simple* is defined. Let $u\mathcal{DS}$ and $su\mathcal{DS}$ be the symbols for these both classes respectively.

It is easy to see that uniformly dense simple implies dense simple, since the definitions can be restricted to the class of monotone funtions. For the classes $su\mathcal{DS}$ and $s\mathcal{DS}$ this inclusion does not hold.

Since there exists a dense simple set A with

$$(\exists^{\infty} n)(n \in \overline{A} \wedge n+1 \in \overline{A}) \qquad ,$$

$u\mathcal{DS}$ is a proper subclass of $\mathcal{DS}$.

Theorem 2.2.3.2 (Dëgtev)/Dë,71/: If M is a maximal set, then M is strongly uniformly dense simple.

Thus we see that for every partial recursive function f the set $\{x : f(x)\downarrow \wedge (x,f(x)] \subseteq M\}$ includes $\overline{M}$ (mod Fin). This property is oftenly used in proofs of Theorems on maximal sets.

Lemma 2.2.3.3: Let $M_1,\dots,M_n$ be maximal sets. Then $M_1 \cap \dots \cap M_n$ belongs to $su\mathcal{DS}$ iff all M_i are pairwise not $\equiv^*$- equivalent.

This Lemma says that

$$Q\mathcal{M} \cap su\mathcal{DS} = Q\mathcal{M}_{\equiv^*} \qquad .$$

Corollary 2.2.3.4: The class $u\mathcal{DS}^{o}$ is closed upwards, but no filter

Proof: Since $\underline{Max} \subseteq u\mathcal{DS}$, $Q\mathcal{M}^{o}$ is the smallest filter including $\underline{Max}$ and $Q\mathcal{M} \not\subseteq u\mathcal{DS}$, the class cannot be a filter in $\mathcal{E}$.

Theorem 2.2.3.5: The classes su$\mathcal{D}\mathcal{S}$ and u$\mathcal{D}\mathcal{S}$ do not coincide.

2.2.4 Hypersimple sets

A function f *majorizes the function g* (*majorizes the set X*) if for almost all n hold $g(n) \leq f(n)$ ($p_X(n) \leq f(n)$) .

Definition 2.2.4.1(Post): A set X is called *hyperimmune* (shortly : h-immune) if X is infinite and X cannot be majorized by any recursive function.

A set is called *hypersimple* (shortly , h-simple) if A is r.e. and $\bar{A}$ is h-immune .

Immediately from the definition follows that dense immune implies h-immune and thus that dense simple implies h-simple. Further h-simple sets are simple.

Denote by $\mathcal{H}\mathcal{S}$ the class of h-simple sets. $\mathcal{H}\mathcal{S}^o$ is a filter in $\mathcal{E}$ and cofinal. This filter is not lattice definable in $\mathcal{E}$, see for this 3.2.2 .

Remark 2.2.4.2. Observe that Post's simple set is not h-simple , since the function $f(e) = 2 \cdot e$ majorizes the complement of this set. Thus $\mathcal{H}\mathcal{S} \neq \mathcal{S}$.

Characterization of the h-simple sets

The h-simple sets can be characterized by using special recursive arrays. Just by meaning of this characterization many facts about $\mathcal{H}\mathcal{S}$ can be proved.

Canonical indizes of finite sets

The finite set $\{x_1,\ldots,x_n\}$ can be assigned the number $2^{x_1}+\ldots+2^{x_n}$. This number is called *canonical index of the set* $\{x_1,\ldots,x_n\}$. (0 is the index of $\emptyset$) .

It is obvious that this is an effective and bijective assignment between all finite sets and ω . With D_x is denoted the finite set with the canonical index x. Obviously $(D_x)_{x\geqslant 0}$ is a recursive array.

A recursive array $(S_n)_{n\geqslant 0}$ of finite sets is called <u>strongly</u> if there is a recursive function f s.t. $S_n = D_{f(n)}$, $n \geqslant 0$.

<u>Theorem 2.2.4.3 (Kusnecov, Medvejev, Uspetski) /Ro, p. 182/</u>: A set X is h-immune iff X is infinite and there is no strongly recursive array $(S_n)_{n\geqslant 0}$ of disjoint sets s.t. $S_n \cap X \neq \emptyset$ for all n .

Thus an r.e. set A is h-simple iff A is coinfinite and there is no strongly recursive array $(S_n)_{n\geqslant 0}$ of disjoint sets with $S_n \cap A \neq \emptyset$ for all n.

2.2.5 Strongly and finitely strongly hypersimple sets

The notion of h-simple set can be still intensified if we take a greater class of recursive arrays as the class of all strongly recursive arrays.

<u>Definition 2.2.5.1</u>: A coinfinite r.e. set A is called <u>strongly hypersimple</u> if there is no recursive array $(S_n)_{n\geqslant 0}$ of disjoint sets with $\bigcup_{n\geqslant 0} S_n = \omega$ and

$$(2.2) \qquad (\forall n)(\, S_n \cap \bar{A} \neq \emptyset \,) \qquad .$$

A coinfinite r.e. set A is called <u>finitely strongly hypersimple</u> if there is no recursive array $(S_n)_{n\geqslant 0}$ of finite and disjoint sets with $\bigcup_{n\geqslant 0} S_n = \omega$ and (2.2) .

Let $s\mathcal{H}\mathcal{S}$ be the symbol for the class of strongly h-simple sets . From the definition it follows

$$s\mathcal{H}\mathcal{S} \subsetneq fs\mathcal{H}\mathcal{S} \subsetneq \mathcal{H}\mathcal{S}$$

h-simple sets. From the definiton it follows

$$s\mathcal{HS} \subseteq fs\mathcal{HS} \subseteq \mathcal{HS} \quad .$$

Further it can be shown that both inclusions are properly.

Remark. In particular the class $fs\mathcal{HS}$ becames always more important inside the analyse of the simple sets. Just with this class many notions can be compared , i.e. either every set from $fs\mathcal{HS}$ has this property or no element satisfies this. We shall see this oftenly in the following.

An example of that what is said in the above Remark is the following Lemma. This Lemma is an application of Lemma 2.1.2 .

Lemma 2.2.5.2: If A is from $fs\mathcal{HS}$ and $(S_n)_{n \geq 0}$ is a recursive array of disjoint finite sets with $\bigcup S_n = \omega$ then there is a recursive function g , s.t.

$$(\forall a.a.n)(\forall x)(\ x \in \bar{A} \cap S_n \longrightarrow x \leq g(n)) \quad .$$

This means that the elements of $\bar{A} \cap S_n$ are bounded by $g(n)$ for almost all n .

2.3 Properties of the maximal sets , a classification of the class of simple sets .

In this point we will continue the subdivision of the class of simple sets under some points of view. It will be regarded here such properties which were investigated in several papers and in particulary applied to the analysing the maximal sets. At the end of this point we give still a classification of all subclasses of simple sets which were defined until now in this paper .

2.3.1 Fullness of sets , full r.e. sets

We consider here the r.e. sets under a point of view which was indroduced in /Ber,Cr/ . This will be make a connection between the r.e. sets and a basic notion of the analysis.

Let X be a subset of ω . Let $l(X)$ be the limit for $n \longrightarrow \infty$ if he exists , of the sequence

$$\frac{\operatorname{card}(X \cap [0,n))}{n} .$$

It is obvious that $l(X)$ if it exists is a real number from $[0,1]$.

We have

- There are r.e. sets X , s.t. $l(X)$ does not exist
- If $X \subseteq Y$ and $l(X)$, $l(Y)$ both exist , then $l(X) \leq l(Y)$
- If $l(X)$ exists then $l(\overline{X})$ exists and $l(X) + l(\overline{X}) = 1$.

Lemma 2.3.1.1: Let A be a dense simple set. Then $l(A)$ exists and is equal to 1 .

Corollary 2.3.1.2(Bernardi/Crociani): If M is a maximal set, then $l(M)$ exists and is equal to 1 .

Remark2.3.1.3. In /Ber,Cr/ it is shown that for every rational number r from $[0,1]$ there is a simple set S with $l(S) = r$. By an easy modification of Post's simple set construction this result can by generalized to all recursive real numbers in $[0,1]$.

The simple sets with the limit equal to 1 and 0 will be still regarded in particulary .

Full r.e. sets

We call a r.e. set X full if $l(X)$ exists and is equal to 1. It is easy to see that there are nonsimple r.e. sets, even coinfinite recursive sets which are full. But we consider here only the full simple sets. Denote with $\mathcal{V}$ the class of all full simple sets.

Lemma 2.3.1.4: The class $\mathcal{V}^0$ is a filter in $\mathcal{E}$

This filter is not cofinal in $\mathcal{E}$. (This can be shown by the same method as later Corollary 8.1.2). From 2.3.1.1 it follows that $\infty\mathcal{S} \leq \mathcal{V}$.

The class $I_{1=0}$

Denote with $I_{1=0}$ the class of the simple sets having the Limit equal to 0. From Remark 2.3.1.3 it follows that $I_{1=0}$ is not empty. Easy can be shown that

Lemma 2.3.1.5: The class $I_{1=0}$ forms an (proper) ideal in $\mathcal{S}$.

Since $\mathcal{S}$ has no lattice definable ideal (except itself) and usually the known subclasses of simple sets if are not filters , then they are at least closed upwards, the family $I_{1=0}$ throughout having a quite easy definition is of some interest.

2.3.2 Monotone and 1-1 sets

A possibility to classify the subsets of ω is the beheviour of the recursive functions on them , e.g. if all recursive functions are monotone or one-one on them. This topic was investigated above all in /Mad,Rob/ .

Definition 2.3.2.1: Let X be an infinite subset of ω . X is called monotone (one-one , shortly 1-1) if for every recursive function f

$$(2.3)\quad \begin{array}{l} (\exists x)(\forall y_1 > x)(\forall y_2 > x)(\ y_1 \leq y_2 \longrightarrow f(y_1) \leq f(y_2)\ ) \\ (\ (\exists x)(\forall y_1 > x)(\forall y_2 > x)(\ y_1 \neq y_2 \longrightarrow f(y_1) \neq f(y_2))\ ) \end{array}$$

or f is constant (mod Fin) on X (i.e. $\forall x > x_0 (x \in X \longrightarrow f(x) = y_0)$, for some x_0, y_0).

Characterization of the monotone and 1-1 sets

To characterize these two notions we need a generalization of the notion cohesive. An infinite subset X of ω is called r-cohesive if

$$(\forall R \text{ rec.})(R \cap X =^* \emptyset \vee X \subset^* R) \quad .$$

A set A is called r-maximal if A is r.e. and $\bar{A}$ is r-cohesive.

Lemma 2.3.2.2: If X is monotone or 1-1 , then X is r-cohesive and dense immune.

Corollary 2.3.2.3: R.e. sets which are co-monotone or co-1-1 are simultaneously r-maximal and dense simple.

The notions monotone set and 1-1 set in general are incomparable, see 10.2 . But if we still require to be co-r.e. we get

Theorem 2.3.2.4(Madan/Robinson): A r.e. set A is co-monotone iff A is co-1-1 .

Lemma 2.3.2.5(Owings): If M is a maximal set, then M is co-monotone (and thus by 2.3.2.4 also co-1-1).

Remark. The inversion of Lemma 2.3.2.5 does not hold as it is shown in /Mad,Rob/, see later 10.2 .
The notions monotone and 1-1 can be intensified in the usual way by taking all partial recursive functions instead of the total recursive functions in 2.3.2.1 .
In place of " $y_1 \leq y_2$ " we put " $y_1 \leq y_2 \wedge f(y_1)\downarrow \wedge f(y_2)\downarrow$" in (2.3) and call these sets strongly monotone and strongly 1-1.
Then it holds

Lemma 2.3.2.6: Let X be a coinfinite r.e. set. The following conditions are equivalent :

1) X is maximal
2) X is strongly co-monotone
3) X is strongly co-1-1 .

Thus we see that the above Lemma includes equivalent properties that a coinfinite r.e. set is maximal.

The lattice $\mathcal{E}_{II}$.

Denote with $\mathcal{E}_{II}$ the family of all 2-ary r.e. relations together with the implications as basic relation between them. Of course $\mathcal{E}_{II}$ is isomorphic with $\mathcal{E}$ (by the mapping : $R(x,y) \longrightarrow \langle x,y\rangle \in \tilde{R}$) . But we use $\mathcal{E}_{II}$, since some states of affairs can be better represented in $\mathcal{E}_{II}$ as in $\mathcal{E}$ (We use $\mathcal{E}_{II}$ later in 5.4) .

Let $R \in \mathcal{E}_{II}$ and $i \in \omega$. Then $(R)_i$ means the set of x with $R(x,i)$.
$\omega \times i$ denotes the R from $\mathcal{E}_{II}$ with $R(x,y) \longleftrightarrow y=i$.
A relation R is called column if R is $\omega \times i$ for some i .

Let M be a maximal relation in $\mathcal{E}_{II}$ (a relation corresponding to a maximal set in $\mathcal{E}$) . From 2.1.1 it follows that either $\overline{M}$ is included in some $\omega \times i$ (mod Fin) or for almost all i we have $|\omega \times i \wedge \overline{M}| \leq 1$. In the second case we can define a partial function $\varphi_{\overline{M}}$, by $\varphi_{\overline{M}}(i) = x$ if $(x,i) \in \overline{M}$. $\varphi_{\overline{M}}$ is defined for all i for which $|\omega \times i \wedge \overline{M}| = 1$.

Lemma 2.3.2.7: 1) $\varphi_{\overline{M}}$ is either constant (mod Fin) or strongly monotone increasing
2) There is a (total) recursive function f with $\varphi_{\overline{M}} \subseteq f$ (i.e. $\varphi_{\overline{M}}(i)\downarrow \wedge \varphi_{\overline{M}}(i)=j . \longrightarrow f(i) = j$) .

2.3.3 Effectively nowhere simple sets

In /Mi,Re/ the notion of so-called nowhere simple set introduced in /Sh/ among others was connected also with the maximal sets. After a small introduction we give here this connection.

Definition 2.3.3.1(Shore): A r.e. set A is called nowhere simple if

$$(\forall e)(\ W_e \setminus A \text{ - infinite} \longrightarrow (\exists f)(\ W_f \text{ - infinite} \wedge W_f \subseteq W_e \setminus A\ )\quad)$$

and effectively nowhere simple if there is a recursive function f with

$$(\forall e)(\ W_{f(e)} \subseteq W_e \setminus A \wedge (\ W_e \setminus A \text{ - infinite} \longrightarrow W_{f(e)} \text{ is infinite})\ ).$$

We see directly from the definition that all recursive sets are nowhere simple and that simple sets are not nowhere simple. That there are also nonrecursive r.e. sets which are nowhere simple follows easily from the following Theorem :

Let A be an nonrecursive r.e. set. A pair (A_1,A_2) of r.e. sets is called splitting of A if both are nonrecursive , disjoint and $A_1 \vee A_2 = A$.

Theorem 2.3.3.2(Friedberg splitting Theorem)/Ro,p.294/: Let A be a nonrecursive r.e. set. Then there is a splitting (A_1,A_2) of A, s.t.

$$(\forall e)(\ W_e \setminus A \text{ not r.e.} \longrightarrow (W_e \setminus A_1 \neq \emptyset \wedge W_e \setminus A_2 \neq \emptyset\ )).$$

Corollary 2.3.3.3(Shore): There are nonrecursive r.e. sets which are nowhere simple.

Proof: Let A be a nonrecursive r.e. set and (A_1,A_2) be a splitting of A as in Theorem 2.3.3.2 (so-called Friedberg splitting of A) . If $W_e \setminus A_1$ is infinite and r.e. take

$W_f = W_e \setminus A$. If $W_e \setminus A_1$ is not r.e. then $W_e \cap A_2$ must be infinite. Thus $W_f = W_e \cap A_2$ satisfies the definiton . Hence A_1 (as also A_2) is a nonrecursive nowhere simple set.

From 2.3.3.3. we get the following definition : Let A be a r.e. set. A set B is called witness set for A if B is r.e. , $A \cap B = \emptyset$ and

$$(\forall e)(\ W_e \setminus A \text{ is infinite} \longrightarrow W_e \cap B \text{ is infinite }) \ .$$

In /Mi,Re/ it is shown that a r.e. set A is effectively nowhere simple iff the set A has a witness set for A . By using this we get

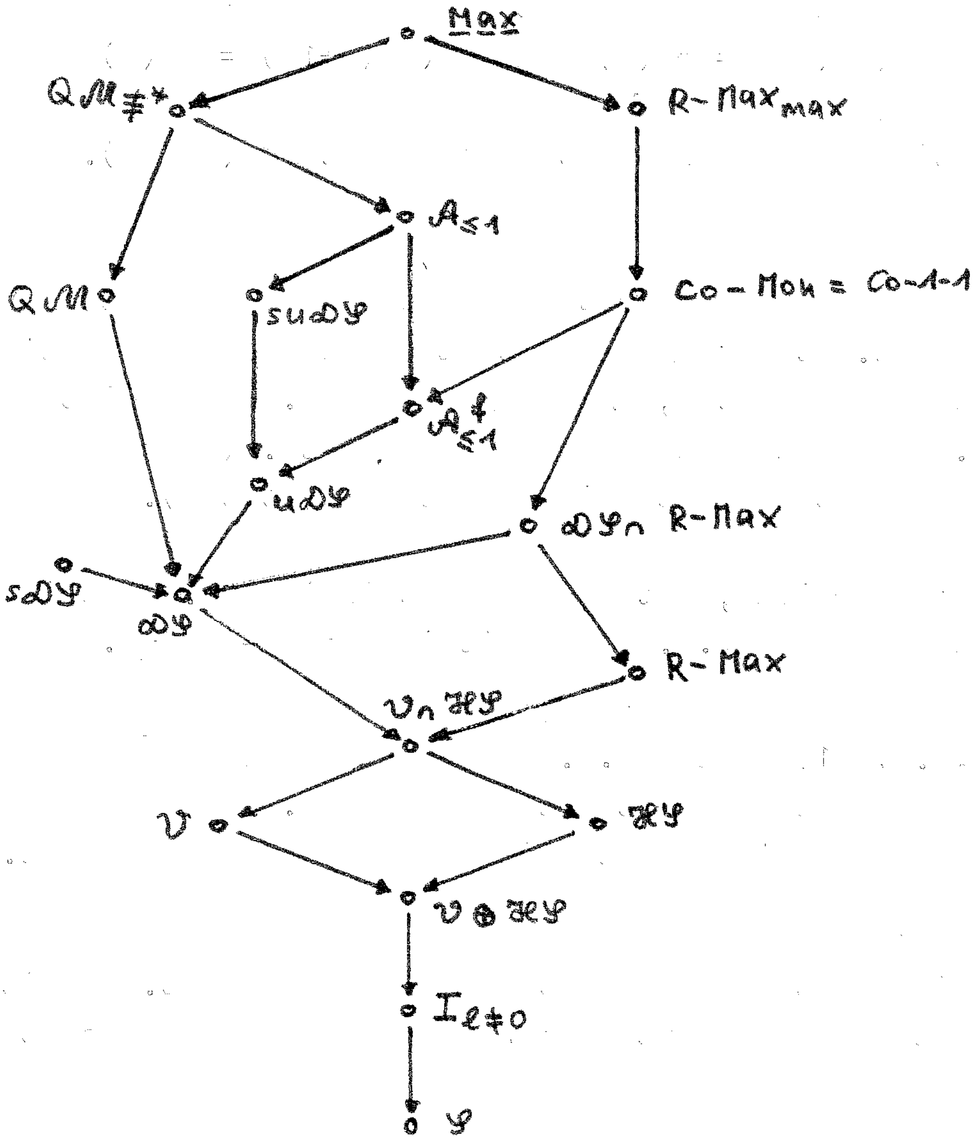
Lemma 2.3.3.4(Miller,Remmel): A simple set M is maximal iff every splitting (M_1, M_2) of M consists of effectively nowhere simple sets M_1 amd M_2 .

2.3.4 A classification of the class of simple sets

The different subclasses of simple sets introducted in 2.2 and 2.3 will be regarded here together. In particular we give the mutual inclusion relations between these classes .

For the following scheme we need further symbols and notions. Let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be two classes of sets. We write $\mathfrak{X}_1 \longrightarrow \mathfrak{X}_2$ if $\mathfrak{X}_1$ is a proper subclass of $\mathfrak{X}_2$. If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are two filters then $\mathfrak{F}_1 \oplus \mathfrak{F}_2$ denotes the smallest filter including both .
Let $I_{1\neq 0}$ mean the class $\{ S \text{ simple} : S \notin I_{1=0} \}$ and $\text{R-Max}_{\max}$ equal to $\{ S : S \text{ is r-maximal} \wedge (\exists M)(\ M \text{ is maximal} \wedge S \subsetneq M\ ) \}$.
With Co-Mon (Co-1-1) we denote the class of coinfinite r.e. sets having a monotone complement (an 1-1 complement) .

The following scheme holds between the subclasses of $\mathcal{S}$ introduced in the subpoints 2.2 and 2.3 :



In the above classification no further implications hold and it can be shown that all inclusions are proper with the exception if $R\text{-}Max_{max}$ is a proper subclass of Co-Mon /Mad,Rob/,
$su\mathcal{D}\mathcal{S} \subsetneq u\mathcal{D}\mathcal{S}$ and $su\mathcal{D}\mathcal{S} \supseteq R\text{-}Max_{max}$.

2.4 Retraceable and regressive sets . An other connection between the complements of coinfinite r.e. sets and recursive functions will be given in this point.

Retraceable sets

We say that a set A is <u>retraceable</u> if there is a partial recursive function f such that f is defined for all elements of A and $f(p_A(0)) = p_A(0)$ and $f(p_A(n+1)) = p_A(n)$ for all $n \geqslant 0$.
The function f is called retracing function (for A).

Obviously all recursive sets are retraceable and other r.e. sets are not. It can be shown that there are even Continuum many retraceable sets , see /Ro/, p. 205 . We are interest in the r.e. sets with retraceable complements. (In this case we can assume that the retracing function is total).

Every nonrecursive , retraceable set is immune. If this set is additionally co-r.e. , then it is even h-immune.

<u>Lemma 2.4.1</u>: 1) If the r.e. set A has a retraceable complement then A belongs to $\mathfrak{H}\mathfrak{Y}$ but not to fs $\mathfrak{H}\mathfrak{Y}$.
2) There are h-simple sets with retraceable complement.

Thus maximal sets have no retraceable complement. But it holds still a stronger property which we give at the end of this point.

Regressive sets

A set A is called <u>regressive</u> if there is a sequence without repetition $(a_n)_{n \geqslant 0}$ of A and a partial recursive function f with $A \subseteq \operatorname{dom}(f)$ such that $f(a_0) = a_0$ and $f(a_{n+1}) = a_n$ for all $n \geqslant 0$.

Directly from the definition it follows that every retraceable set also is regressive (by taking $(p_A(n))_{n \geqslant 0}$ for $(a_n)_{n \geqslant 0}$). Since all r.e. sets are regressive , these both notions are different. But also if the sets are co-r.e. , regressive is unequal to retraceable , see /Mc,66/ .

Between retraceable and regressive sets which are co-r.e. the following connection holds :

Theorem 2.4.2 (Dekker)/De/: Let A be a r.e. set and co-regressive . Then there is a recursive permutation ρ such that $\rho(A)$ is co-retraceable .

Corollary 2.4.3: The r.e. sets which are co-regressive form the closure of the r.e. sets with retraceable complement under recursive permutations.

The r.e. sets which are co-regressive are also not from fs $\mathcal{HS}$. Thus

$$\text{R.e.,co-retr.} \xrightarrow[\equiv\text{-closure}]{} \text{R.e.,co-regr.} \longrightarrow (\mathcal{HS} \setminus \text{fs}\,\mathcal{HS})$$

and both implications are properly.

Retraceable and regressive subsets

The sets which have retraceable or regressive subsets have also interesting properties. To this the following facts :

1) A set has an infinite retraceable subsets iff it has an infinite regressive subset.

2) The assertion " X has an infinite retraceable subset " is recursively invariant. Since " X is co-r.e. " also is it , the intersection of both is recursively invariant .

3) If M is maximal then $\overline{M}$ does not contain an infinite retraceable subset.

Remarks. The property 3) will be still used later in point 5. The sets with infinite , retraceable subsets can be characterized in an other equivalent form, see /Ro/,p.321 . The same holds if the sets are additionally co-r.e. .

That the complements of the maximal sets do not include infinite retaceable subsets follows from a stronger property which can be shown by applying 2.1.2 and 2.2.3.2 .

Theorem 2.4.4 : If M is a maximal set and f a recursive function then

$$(\exists^{\infty} n)(\ f(p_{\bar{M}}(n+1) \leq p_{\bar{M}}(n) \longrightarrow f|\bar{M} \text{ is constant } (\mathrm{mod\ Fin})).$$

Thus if $X = \{x_0 < x_1 < \ldots\}$ is an infinite , retraceable subset of $\bar{M}$ and f a retracing function for X then for infinitely many n

$$f(p_X(n+1)) = p_X(n) = p_{\bar{M}}(m) < p_{\bar{M}}(m+1) \leq p_X(n+1) = p_{\bar{M}}(l+1)$$

for some m and l with $m \leq l$. Thus

$$f(p_M(l+1)) \leq p_{\bar{M}}(m) \leq p_{\bar{M}}(l) \quad .$$

Hence $f|\bar{M}$ is constant (mod Fin) , what is not possible.

2.5 Recursive arrays and enumeration order. Let M be a maximal set and g an simultaneous enumeration of $(W_e)_{e \geq 0}$. Then by using g it can be defined a linear order inside the set $\mathrm{Cof}_M = \{ e : W_e \cup M =^* \omega \}$. For numbers e and f from Cof_M let $e <_{g,M} f$ holds if

$$(\forall a.a.x)(\ x \in \bar{M} \longrightarrow x \text{ is earlier enumerated in } W_e \text{ as in } W_f \text{ by } g\ ) \ .$$

What can be said about the possible order types of $<_{g,M}$? Known is only that there are maximal sets M , s.t. for every g $<_{g,M}$ does not coincide with $<$ (the order of natural numbers) in Cof_M , see for this 3.3 . Exists a maximal set M and an enumeration g such that $<_{g,M}$ is equal to $<$ in Cof_M ?

Within this topic we have the following Theorem which will be used later for automorphism constructions

Theorem 2.5.1(Soare)/So,74/: Let C be a coinfinite r.e. set. Then there is a recursive array $(Z_n)_{n\geq 0}$ with $Z_0 = C$ and

$$(\forall X \text{ r.e})(X \subset^* C \vee X \cup C = \omega . \longrightarrow (\exists n)(X =^* Z_n))$$

and a simultaneous enumeration g of $(Z_n)_{n\geq 0}$, s.t.

(2.4) $(\forall n)(Z_n -_g Z_0$ is infinite $\longrightarrow \bar{C} \subseteq Z_n)$

(2.5) $(\forall n)(\forall m)(n<m \wedge \bar{C} \subseteq Z_n \wedge \bar{C} \subseteq Z_m . \longrightarrow$
$(\forall \text{a.a.}x)(x \in \bar{C} \longrightarrow x \in Z_n -_g Z_m))$.

Remarks.

1) We call a recursive array $(X_n)_{n\geq 0}$ $=^*$- complete or skeleton if $(\forall e)(\forall n)(W_e =^* X_n)$. Observe that the array $(Z_n)_{n\geq 0}$ in Theorem 2.5.1 is $=^*$- complete if C is a maximal set.

2) From Theorem 2.5.1 follows that for every maximal set M there is an enumeration g , s.t. $<_{g,M}$ includes the order type ω . This holds , since $(Z_n)_{n\geq 0}$ can be recursively embedded into $(W_e)_{e\geq 0}$ and so such an enumeration g (preserving (2.5) for Z_n) can be found.

This considerations can be provided also only for $(W_e)_{e\geq 0}$.

Lemma 2.5.2: There is no simultaneous enumeration g of $(W_e)_{e\geq 0}$ s.t.

$$(\forall e)(\forall f)(W_e \text{ and } W_f \text{ both infinite} \wedge e<f. \longrightarrow (\forall \text{a.a.}s)(|W_{f,s}| \leq |W_{e,s}|)) .$$

Proof: Since $(W_e)_{e \geq o}$ is a standard array there is a recursive function f , strinctly monotone and all $W_{f(n)}$, $n \geq o$ are infinite. If the Lemma wouldn't be true, then we would have

$$W_e \text{ is infinite} \longleftrightarrow (\exists s_o)(\forall s \geq s_o)(|W_{f(e),s}| \leq |W_{e,s}|).$$

Hence $\{e : W_e \text{ is infinite}\} \in \Sigma_2^o$. This cannot be , since this set is Π_2^o - complete .

Exists a simultaneous enumeration of $(W_e)_{e \geq o}$, s.t. the order $(\forall \text{ a.a.s})(|W_{f,s}| \leq |W_{e,s}|)$ is a linear order in $\{e: W_e \text{ is infinite}\}$?

Remark. By using the method of Theorem 2.5.1 we can construct a skeleton with a simultaneous enumeration of this array , s.t. the order even is equal to $<$ inside the set of indizes sets .

2.6 Speedable and levelable sets .

There is still an other topic of investigations of the connections between the maximal sets and simultaneous enumerations of recursive arrays.
Let X be a r.e. set (in particulary a maximal set). We will here compare the sets $W_{i,s}$ and $W_{j,s}$ with $W_i = W_j = X$, where these sets are enumerated by some simultaneous enumeration. To this there will be regarded here two problems investigated above all in /So,77/ .

Speedable sets

At first it will be investigated the set $W_{i,s} \setminus W_{j,s}$ for r.e. sets W_i , W_j with $W_i = W_j$.
Let g be a simultaneous enumeration of $(W_e)_{e \geq o}$. With Φ_g we denote the partial recursive function defined by

$$\Phi_g(i,x) \simeq y \longleftrightarrow g(y) = \langle x,i \rangle .$$

For numbers i and j with $W_i = W_j$ and every number x we have $\Phi_g(i,x)\downarrow$ iff $\Phi_g(j,x)\downarrow$.

<u>Definition 2.6.1(Blum)</u>: A r.e. set A is called <u>speedable</u> if

$$(2.6)\qquad (\forall i)(\ A = W_i \longrightarrow (\forall h \text{ rec.fct})(\exists j)(\ A = W_j \wedge (\exists^\infty x \in A)(\ \Phi_g(i,x) > h(x, \Phi_g(j,x))\ )\ \ )\ .$$

<u>Remarks 2.6.2</u> . 1) The definition 2.6.1 depends from the simultaneous enumeration g. But from the next Theorem it follows that the used enumeration g is not essential .
2) If we take $h(x,y) = y$ then we get $\Phi_g(j,x) < \Phi_g(i,x)$. This means that infinitely many $x \in A$ are earlier enumerated in W_j as in W_i . But since h can be arbitrary , this means that infinitely many $x \in A$ are enumerated much more earlier in $W_{\cancel{i}\,j}$ as in $W_{\cancel{j}\,i}$.

<u>Theorem 2.6.3(Soare)/So,77/</u>: A r.e. set A is nonspeedable (resp. to a simultaneous enumeration g) iff $\{e : W_e \cap \bar{A} \neq \emptyset\} \in \Delta^0_2$.

<u>Remarks 2.6.4</u>. 1) Since the property $\{e : W_e \cap \bar{A} \neq \emptyset\} \in \Delta^0_2$ is independend from the enumeration , the notion of speedable set also is it.
2) If A is not speedable then the index i and the function h depend from g .

In point 8 we characterize the r.e. sets A with $\{e : W_e \cap \bar{A} \neq \emptyset\} \in \Delta^0_2$. It can be shown that all sets from fs $\mathcal{H}\mathcal{S}$ have not this property . Thus all sets from fs $\mathcal{H}\mathcal{S}$ are speedable .

<u>Corollary 2.6.5(Marques)</u>: Maximal sets are speedable .

In oposite to 2.6.5 there are also even simple sets which are nonspeedable .

In the following Lemma there is given an equivalent condition for to be nonspeedable.

Lemma 2.6.6(Blum,Marques): A r.e. set A is nonspeedable iff there is a recursive function f s.t. for all j

$$W_j \cap \bar{A} = W_{f(j)} \cap \bar{A}$$
$$W_j \subseteq A \longrightarrow W_{f(j)} \text{ is finite} \quad .$$

Levelable sets

Suppose W_i is not recursive and g is a simultaneous enumeration. Then for every recursive function h there are infinitely many $x \in W_i$ which are enumerated in W_i in a step which cames after $h(x)$. But the set of these x depends from i . For an other index j (with $W_j = W_i$) infinitely many x from the set above can be enumerated in a step before $h(x)$. This will be investigated here.

Definition 2.6.7(Blum): A r.e. set A is called levelable if

$$(\exists f \text{ rec. fct})(\forall i)(W_i = A \longrightarrow (\forall h \text{ rec. fct})(\exists j) \\ (W_j = A \wedge (\exists^{\infty} x \in A)(\Phi_g(i,x) > h(x) \wedge \\ \Phi_g(j,x) \leq f(x)))) \; .$$

In terms of special recursive arrays the levelable sets can be characterized.

A recursive array $(R_n)_{n \geq 0}$ of recursive sets is uniformly if $\{(i,x) : x \in R_i \; , \; i \geq 0\}$ is a recursive relation.

A recursive array $(R_n)_{n \geq 0}$ of recursive sets is cofinal in A (A a r.e. set) if

$$(\forall R \text{ rec.})(\bar{A} \subseteq R \longrightarrow (\exists n)(R_n \subseteq A \wedge R_n \cap R \text{ is infinite})) \; .$$

Theorem 2.6.8(Blum,Marques,Morris)/So,77/: A r.e. set A is levelable iff there is a recursive array $(R_n)_{n\geqslant 0}$ of recursive sets uniformily and cofinal in A .

In particulary from this Theorem follows that the notion of levelable set is independend from the simultaneous enumeration of $(W_e)_{e\geqslant 0}$.

Theorem 2.6.9(Soare): A r.e. set A is nonlevelable iff $\{n : R_n \cap \bar{A} \neq \emptyset\} \in \Delta_2^0$.

Corollary 2.6.10: Levelable sets are speedable .

Theorem 2.6.11(Soare): Every set from fs $\mathcal{HS}$ is levelable .

Corollary 2.6.12(Blum,Marques): Every maximal set is levelable

Thus we have the following sequence of inclusions :

$$\boxed{\underline{\text{Max}} \subseteq \text{fs}\,\mathcal{HS} \subseteq \text{levelable} \subseteq \text{speedable}} \quad .$$

3. AUTOMORPHISMS OF THE LATTICE OF RECURSIVELY ENUMERABLE SETS AND MAXIMAL SETS

The class of maximal sets is obviously definable in $\mathcal{E}$. Is this class decomposable into nonempty subclasses by using only the lattice operations ?

This question of Lachlan from 1968 in /La,68a/ was answered by Soare 1974 in /So,74/. This family of r.e. sets cannot be further subdivided inside the lattice of recursively enumerable sets. This means that two arbitrary maximal sets are automorhic in $\mathcal{E}$. The proof of this is the main topic of this point.

For showing this automorphisms result it was necessary to develop a new technique of automorphism construction. By this new method it can be shown that also r.e. sets are automorphic which are not recursively invariant (mod Fin).

The proof is quite extensive and technical expensive. For these it is necessary to construct recursive arrays with special properties.

It will be given here only a survey of the automorphisms construction from which the idea of the proof will be became clear.

Besides this in this point there are still included further results about automorphism properties of the lattice $\mathcal{E}$ and above all corollaries which follows from the result that all maximal sets are automorphic.

The main literature which was used for this point ~~was~~ is the paper /So,74/.

3.1 Basic definitions and notions about automorphisms of the lattice $\mathcal{E}$.

3.2 A hierarchy of automorphisms of $\mathcal{E}$, effective automorphisms

3.3 Automorphism between maximal sets

3.4 Corollaries and remarks to the automorphism proof

3.1 Basic definitions and notions about automorphisms of the lattice $\mathcal{E}$. We give at first in this subpoint the basic definitions and notions for the automorphism analyse of $\mathcal{E}$. Further here are included general properties of countable sublattices of $\mathcal{P}(\omega)$ under the view of points of the automorphism analyse of them.

Definition 3.1.1: Let $\mathcal{L}_1 = (L_1, \wedge_1, \vee_1)$ and $\mathcal{L}_2 = (L_2, \wedge_2, \vee_2)$ be lattices . A mapping Φ from L_1 in L_2 is called isomorphism if Φ is bijective and for elements x_1, x_2, x_3 from L_1 hold

$$\begin{aligned} & x_1 \wedge_1 x_2 = x_3 \longleftrightarrow \Phi(x_1) \wedge_2 \Phi(x_2) = \Phi(x_3) \\ & x_1 \vee_1 x_2 = x_3 \longleftrightarrow \Phi(x_1) \wedge_2 \Phi(x_2) = \Phi(x_3) \quad . \end{aligned} \tag{3.1}$$

If Φ is a mapping from L_1 into L_2 for which (3.1) holds with $\longrightarrow$ in place of $\longleftrightarrow$ then Φ is called homomorphism .

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic structures then we write shortly $\mathcal{L}_1 \cong \mathcal{L}_2$.
With $\mathrm{Iso}(\mathcal{L}_1, \mathcal{L}_2)$ we denote the family of all isomorphisms between $\mathcal{L}_1$ and $\mathcal{L}_2$ and with $\mathrm{Aut}(\mathcal{L})$ the group of automorphisms of $\mathcal{L}$ (i.e. $\mathrm{Aut}(\mathcal{L}) = \mathrm{Iso}(\mathcal{L}, \mathcal{L})$) .

If for elements x_1, x_2 from $\mathcal{L}$ there is an $\Phi \in \mathrm{Aut}(\mathcal{L})$ with $\Phi(x_1) = x_2$ we write $x_1 \cong_{\mathcal{L}} x_2$ (or if $\mathcal{L}$ is known only $x_1 \cong x_2$) .
Let $x \in \mathcal{L}$. The set

$$\{ y \in \mathcal{L} : \ (\exists \Phi \in \mathrm{Aut}(\mathcal{L}))(\Phi(x) = y) \}$$

is called the orbit of x in $\mathcal{L}$ and is denoted with $\mathcal{O}_{\mathcal{L}}(x)$. Hence an orbit consists of such elements of $\mathcal{L}$ which inside the lattice cannot be distinguished.

A subfamily $\mathfrak{X}$ of $\mathcal{L}$ is called <u>lattice theoretic definable</u> if the set

$$\{ \Phi(x) \; : \; x \in \mathfrak{X} \; , \; \Phi \in \mathrm{Aut}(\mathcal{L}) \}$$

is contained in $\mathfrak{X}$.

We see that $\mathfrak{X}$ is lattice theoretic definable iff $\mathfrak{X}$ is a union of orbits .
The class <u>Max</u> obviously is lattice theoretic definable in $\mathcal{E}$.
Hence here only is interesting of how many orbits this family consists.

<u>Connection between Aut($\mathcal{E}$) and Aut($\mathcal{E}^*$)</u>
Let $\mathcal{L}$ be a sublattice of $\mathcal{P}(\omega)$. $\mathcal{L}^* \leq \mathcal{P}^*(\omega)$ means that $\mathcal{L}$ is closed respectively to $=^*$.
The mapping $\varphi_{kan} : \mathcal{L} \longrightarrow \mathcal{L}^*$ with $\varphi_{kan}(X)^* = X^*$ denotes the canonical homomorphism from $\mathcal{L}$ onto $\mathcal{L}^*$.

<u>Lemma 3.1.2(Lachlan)/La,68/</u>: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be countable sublattices of $\mathcal{P}(\omega)$ with

- $\mathcal{L}_1^* \leq \mathcal{P}^*(\omega)$, $\mathcal{L}_2^* \leq \mathcal{P}^*(\omega)$
- $\emptyset \in \mathcal{L}_1 \longleftrightarrow \emptyset \in \mathcal{L}_2$, $\omega \in \mathcal{L}_1 \longleftrightarrow \omega \in \mathcal{L}_2$.

Then for all $X \in \mathcal{L}_1$

$$(\forall \Psi^* \in \mathrm{Iso}(\mathcal{L}_1^*, \mathcal{L}_2^*))(\exists \Phi \in \mathrm{Iso}(\mathcal{L}_1, \mathcal{L}_2)) \\ (\Psi^*(\varphi^1_{kan}(X)) = \varphi^2_{kan}(\Phi(X)) \;) \; .$$

φ_{kan} also gives a homomorphism from Aut($\mathcal{L}$) into Aut($\mathcal{L}^*$) by

$$(3.2) \qquad \Phi^*(\varphi_{kan}(X)) \; = \; \varphi_{kan}(\Phi(X)) \; .$$

<u>Corollary 3.1.3</u>: The mapping $\Phi \in \mathrm{Aut}(\mathcal{E}) \longrightarrow \Phi^* \in \mathrm{Aut}(\mathcal{E}^*)$ defined in (3.2) is surjective.

Corollary 3.1.4(Soare)/So,74/: If the r.e. sets X_1, X_2 are infinite and coinfinite then

$$X_1 \cong_{\mathcal{E}} X_2 \longleftrightarrow X_1^* \cong_{\mathcal{E}} X_2^* \quad .$$

Corollary 3.1.5: Let $\mathcal{X}$ be a subclass of $\mathcal{E}$ consisting of infinite and coinfinite sets. Then $\mathcal{X}$ is lattice theoretic definable iff $\mathcal{X}^* \subseteq \mathcal{E}^*$ and $\mathcal{X}^*$ is lattice definable in $\mathcal{E}^*$.

Permutations of ω and induced automorphisms

It will be introduced here two ways of representations of automorphisms of the lattice $\mathcal{E}$. Especially the second one is very usefull for the constructions of automorphisms.

Definition 3.1.6: Let $\mathcal{L}^*$ be a sublattice of $\mathcal{P}^*(\omega)$, $\Phi^* \in \mathrm{Aut}(\mathcal{L}^*)$ and p a permutation of ω . We say that p induces the automorphism Φ^* if for all $X^* \in \mathcal{L}^*$ $\Phi^*(X^*) = p(X)^*$ for some $X \in X^*$.

Theorem 3.1.7(Soare)/So,74/: Let $\mathcal{L}$ be a countable sublattice of $\mathcal{P}(\omega)$ and $(X_n)_{n \geq 0}$ and $(Y_n)_{n \geq 0}$ two sequences including all elements of $\mathcal{L}$.
Let $st_1(x,e)$ be the state of length e of x resp. to $(X_n)_{n \geq 0}$ and $st_2(x,e)$ the corresponding state resp. to $(Y_n)_{n \geq 0}$, see the Introduction. The mapping $X_n^* \longrightarrow Y_n^*$ is an automorphism of $\mathcal{L}^*$ iff for every e and state σ with $|\sigma| = e$

$$\{x : st_1(x,e) = \sigma\} \text{ is infinite } \longleftrightarrow$$
$$\{x : st_2(x,e) = \sigma\} \text{ is infinite} \quad .$$

In other words iff

$$X_0^{\sigma(0)} \cap \ldots \cap X_{e-1}^{\sigma(e-1)} \text{ is infinite } \longleftrightarrow$$
$$Y_0^{\sigma(0)} \cap \ldots \cap Y_{e-1}^{\sigma(e-1)} \text{ is infinite} \quad .$$

Corollary 3.1.8(Soare): If $\mathcal{L}$ is a countable sublattice of $\mathfrak{P}(\omega)$ then every Φ^* from $\mathrm{Aut}(\mathcal{L}^*)$ is induced by a permutation of ω.

Recursive permutations

It is obvious that every recursive permutation p of ω induces an automorphism of $\mathcal{E}$, by $X \longrightarrow p(X)$. We call such automorphisms recursive and denote the group of recursive automorphisms with $\mathrm{Aut}_r(\mathcal{E})$. $\mathcal{O}_r(X)$ means the orbit of X resp. to $\mathrm{Aut}_r(\mathcal{E})$.

If φ is a partial recursive and injective function then φ induces an isomorphism between $\mathcal{E}|\mathrm{dom}(\varphi)$ and $\mathcal{E}|\mathrm{rg}(\varphi)$. Thus every sublattice $\mathcal{E}|A$ for infinite r.e. sets A is isomorphic with $\mathcal{E}$ and this isomorphism is induced by a recursive (injective) function. This result will be still improved later in 3.4.

In the lattice $\mathcal{E}$ the following two properties hold :

$$(3.3)\qquad (\forall e)(\ W_e \text{ is infinite} \longrightarrow (\exists R \text{ rec.})(\ R \text{ is infinite} \wedge R \subseteq W_e\ ))$$

$$(3.4)\qquad (\forall R \text{ rec.})(\ R \text{ is infinite} \longrightarrow |\mathrm{Aut}_r(\mathcal{E}|R)^*)| \geq 2\ ) .$$

From these both it can be concluded, see /So,74/ :

Lemma 3.1.9(Lachlan): $\mathrm{Aut}(\mathcal{E}^*)$ has Continuum many elements

Proof: From (3.3) it follows that there is a sequence $(R_i)_{i\geq 0}$ of infinite recursive sets, disjoint and $\bigcup_{i\geq 0} R_i = \omega$, s.t.

$$(\forall e)(\ W_e \subseteq R_0 \cup \ldots \cup R_e \ \vee\ \overline{W_e} \subseteq R_0 \cup \ldots \cup R_e\ ) .$$

Thus if $\Psi_i \in \mathrm{Aut}((\mathcal{E}|R_i))$, $i=0,1,\ldots$ then $\bigcup_{i\geq 0} \Psi_i \in \mathrm{Aut}(\mathcal{E})$. From (3.4) we see that there are Continuum many different such sequences $(\Psi_i)_{i\geq 0}$.

But observe that r.e. sets automorphic by an automorphism of Lemma 3.1.9 are already automorphic by an $\Phi \in \mathrm{Aut}_r(\mathcal{E})$. Thus the orbits resp. to these automorphisms are equal to those formed by $\mathrm{Aut}_r(\mathcal{E})$.

Permutations of ω and presented automorphisms

Now it will be given the second representation of automorphisms of $\mathcal{E}^*$. Just this will be used for the automorphism construction in 3.3 .

Definition 3.1.10: Let h be a permutation of ω and $\Phi^* \in \mathrm{Aut}(\mathcal{E}^*)$. We say that *h presents the automorphism Φ^** if for all n $\Phi^*(W_n^*) = W_{h(n)}^*$.

It is obvious that every automorphism of $\mathcal{E}^*$ is presented by some permutation of ω .

Problem. Exists for automorphic r.e. sets always an arithmetical permutation presenting an automorphism between these both ?

We say that an automorphism *Φ^* is presented by a pair of functions (f,g)* if for all n

$$\Phi^*(W_n^*) = W_{f(n)}^* \qquad \Phi^{*-1}(W_n^*) = W_{g(n)}^* \quad .$$

If the permutation h presents the automorphism Φ^* then also the pair (h,h^{-1}) and converse if the pair (f,g) presents Φ^* then there is a permutation h which presents Φ^* and which can be chosen recursive in $f \oplus g$, see /So,74/ .
Thus , if f and g are from Δ_n^0 then also h belongs to Δ_n^0 .

3.2 A hierarchy of the automorphisms of $\mathcal{E}^*$, effective automorphisms. Inside the group $\mathrm{Aut}(\mathcal{E}^*)$ can be defined a hierarchy of subgroups if we consider to which arithmetical class a permutation presenting the automorphism belongs. This hierarchy will be described here shortly. In particular the automorphisms presented by a recursive permutation are of interest.

Definition 3.2.1: An automorphism Φ^* of $\mathcal{E}^*$ is called effective if Φ^* is presented by a recursive permutation (or equivalent to this by a pair of recursive functions) .

$\mathrm{Aut}_{eff}(\mathcal{E}^*)$ denotes the subgroup of all effective automorphisms of $\mathcal{E}^*$. If there is a Φ^* from $\mathrm{Aut}_{eff}(\mathcal{E}^*)$ with $\Phi^*(X^*) = Y^*$ we write $X \cong_{eff} Y$. In general we denote with $\mathrm{Aut}_{\Delta^0_n}(\mathcal{E}^*)$ the subgroup of all automorphisms presented by a Δ^0_n permutation (Thus $\mathrm{Aut}_{\Delta^0_1}(\mathcal{E}^*) = \mathrm{Aut}_{eff}(\mathcal{E}^*)$) .

We have the following hierarchy :

$$\begin{aligned} \mathrm{Aut}_r(\mathcal{E}^*) \subseteq \mathrm{Aut}_{eff}(\mathcal{E}^*) = \mathrm{Aut}_{\Delta^0_1}(\mathcal{E}^*) \subseteq \mathrm{Aut}_{\Delta^0_2}(\mathcal{E}^*) \subseteq \\ \mathrm{Aut}_{\Delta^0_3}(\mathcal{E}^*) \subseteq \dots \subseteq \mathrm{Aut}_{\Delta^0_7}(\mathcal{E}^*) \subseteq \dots \subseteq \mathrm{Aut}(\mathcal{E}^*) \quad . \end{aligned} \tag{3.5}$$

Remarks to the inclusion chain (3.5)

The first inclusion is properly what follows from the Theorem:

Theorem 3.2.2(Martin)/So,74/: There is a h-simple set H and an effective automorphism Φ^*, s.t. $\Phi^*(H^*)$ is not h-simple (mod Fin) .

Corollary 3.2.3: $\mathrm{Aut}_r(\mathcal{E}^*) \neq \mathrm{Aut}_{eff}(\mathcal{E}^*)$.

Proof: Since the class of h-simple sets is invariant under recursive permutations , recursive and effective automorphisms do not coincide .

An other property in (3.5) follows from the possibility to approximate recursive the Δ^0_2 sets and that we regard automorphisms of $\mathcal{E}^*$ (and not of $\mathcal{E}$).

Lemma 3.2.4(Jockusch)/So,74/: The subgroups $\mathrm{Aut}_{\Delta^0_1}(\mathcal{E}^*)$ and $\mathrm{Aut}_{\Delta^0_2}(\mathcal{E}^*)$ are equal.

Remark 3.2.5. Lemma 3.2.4 cannot be generalized to every n. The reason for this is that not every Σ^0_n sequence of r.e. sets can be embedded into $(W_e)_{e\geq 0}$ by a Σ^0_n function. It can be shown that for every $n \geq 2$ $\mathrm{Aut}_{\Delta^0_n}(\mathcal{E}^*)$ is a proper subgroup of $\mathrm{Aut}_{\Delta^0_{n+1}}(\mathcal{E}^*)$.

The subgroup $\mathrm{Aut}_{\Delta^0_7}(\mathcal{E}^*)$ is of special interest, since by Lemma 3.1.11 ?

$$(\forall X \text{ r.e.})(\forall Y \text{ r.e.})(X^* \cong_{\mathcal{E}^*} Y^* \longleftrightarrow (\exists \Phi^* \in \mathrm{Aut}_{\Delta^0_7}(\mathcal{E}^*))(\Phi^*(X^*) = Y^*)).$$

In point 3.4 we shall see that for maximal sets we can find even a Φ^* from $\mathrm{Aut}_{\Delta^0_3}(\mathcal{E}^*)$ with $\Phi^*(X^*) = Y^*$.

Effective automorphisms and maximal sets

Theorem 3.2.6(Soare): There are maximal sets A and B such that for every effective automorphism Φ^* $\Phi^*(A^*) \neq B^*$.

Since this Theorem can be generalized to arbitrary finite many maximal sets we have

Corollary 3.2.7: The class Max decomposes into infinitely many orbits resp. to $\mathrm{Aut}_{\mathrm{eff}}(\mathcal{E}^*)$.

3.3 Automorphism between maximal sets . In this subpoint we will give the sketch of the proof that two arbitrary maximal sets are automorphic .

Let A and B be maximal sets. We shall show that there is a bijective mapping p from A onto B such that

$$p(W_n \cap A) \text{ and } p^{-1}(W_n \cap B) \text{ are r.e.}$$

and

$$\begin{aligned} \bar{A} \subset^* W_n &\longleftrightarrow p(W_n \cap A) \cup \bar{B} \text{ is r.e.} \\ \bar{B} \subset^* W_n &\longleftrightarrow p^{-1}(W_n \cap B) \cup \bar{A} \text{ is r.e.} \end{aligned}$$

This both insures that A^* and B^* are automorphic and thus also A and B . Important is here that for maximal sets hold $W_e \subset^* A$ or $\bar{A} \subset^* W_e$, analogously for B .

The function p is not constructed directly . This will follows from other results.
We shall construct recursive arrays $(U_n)_{n \geq 0}$, $(V_n)_{n \geq 0}$, $(\hat{U}_n)_{n \geq 0}$, $(\hat{V}_n)_{n \geq 0}$, s.t.

1° $(U_n)_{n \geq 0}$ and $(V_n)_{n \geq 0}$ are skeletons

2° $\bar{A} \subset^* U_n \longleftrightarrow \bar{B} \subset^* \hat{U}_n$ and $\bar{B} \subset^* V_n \longleftrightarrow \bar{A} \subset^* \hat{V}_n$

3° For states $\sigma = (\sigma_0 \sigma_1 \ldots \sigma_{e-1})$ and $\tau = (\tau_0 \tau_1 \ldots \tau_{k-1})$

$$\begin{aligned} &A \cap (\bigcap U(\sigma) \cap \bigcap \hat{V}(\tau)) \text{ is infinite iff} \\ &B \cap (\bigcap \hat{U}(\sigma) \cap \bigcap V(\tau)) \text{ is infinite} \end{aligned}$$

This three properties insure that $A^* \cong_{\mathcal{E}^*} B^*$. The property 3° gives the possibility of defining p , see 3.1.8 .

Remarks 3.3.1 . 1) We cannot take for $(U_n)_{n \geq 0}$ and $(V_n)_{n \geq 0}$ the standard array $(W_e)_{e \geq 0}$, since additionally to be property, to be a skeleton , $(U_n)_{n \geq 0}$ and $(V_n)_{n \geq 0}$ must have further properties (necessary for finding $(\hat{V}_n)_{n\geq o}$ and $(\hat{U}_n)_{n\geq o}$) which $(W_e)_{e\geq o}$ for arbitrary maximal sets does not have). This is the reason why not all maximal sets are effectively automorphic.

2) The property 1^o can be weaken. We say that a array $(X_n)_{n\geq o}$ – generates $\mathcal{E}^*$, if every r.e. set is a combination of elements of $(X_n)_{n\geq o}$ by unions and intersections (mod Fin). It is sufficient to claim that $(U_n)_{n\geq o}$ and $(V_n)_{n\geq o}$ generates $\mathcal{E}^*$.

We shall not work with skeletons but with arrays which generate $\mathcal{E}^*$.

The proof of the automorphism between A and B , i.e. the constructions of the arrays $(U_n)_{n\geq o}$, $(V_n)_{n\geq o}$, $(\hat{V}_n)_{n\geq o}$, $(\hat{U}_n)_{n\geq o}$ is divided into two parts which are formulated as Theorems.

First we give two definitions necessary for the first Theorem.

Let X be a coinfinite r.e. set. A recursive array $(U_n)_{n\geq o}$ is called $\bar{X}$ – complete if for every n U_n is finite or $\bar{X} \subseteq U_n$ and

$$(\forall Y \text{ r.e.})(\bar{X} \subseteq Y \longrightarrow (\exists n)(\bar{X} \subseteq U_n \wedge U_n \subseteq Y)) .$$

Observe that in the case that X is maximal an array $(X_n)_{n\geq o}$ generates $\mathcal{E}^*$ if e.g. $(X_{2n})_{n\geq o}$ is $\bar{X}$-complete and $X_{2n+1} =^* W_n \cap X$, $n\geq o$.

A recursive array $(U_n)_{n\geq o}$ is weak decreasing if

$$(\forall n)(\forall m)(n<m \wedge U_n, U_m \text{ are infinite } . \longrightarrow U_m \subset^* U_n \wedge U_n \setminus U_m \text{ are infinite}).$$

Theorem 3.3.2: Let A and B be two nonrecursive r.e. sets. Then there are recursive arrays $(U_n)_{n\geq o}$ and $(V_n)_{n\geq o}$ and a simultaneous enumeration g of all A , $(U_n)_{n\geq o}$, B , $(V_n)_{n\geq o}$ such that

1^{oo} $(U_n)_{n \geq o}$ is $\bar{A}$ - complete and weak decreasing with
$A \setminus U_n$ infinite for all n

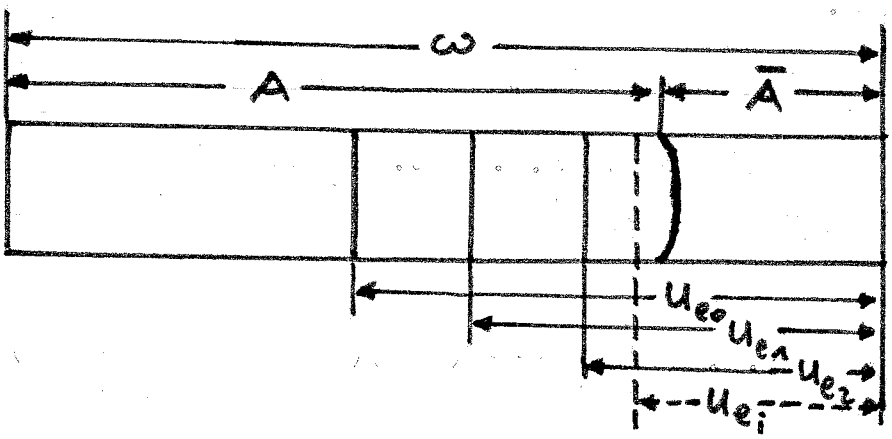
2^{oo} $(V_n)_{n \geq o}$ is $\bar{B}$ - complete and weak decreasing with
$B \setminus V_n$ infinite for all n

3^{oo} $\bar{A} \subset^* U_n \longleftrightarrow \bar{B} \subset^* V_n$ for all n

4^{oo} $(A -_g U_n) \cap U_n = (B -_g V_n) \cap V_n = \emptyset$ for all n .

Remarks. 1. Since we claim that A and B are nonrecursive , both sets are infinite and coinfinite .
2. $A \setminus U_n$ is infinite in 1^{oo} means that no set U_n covers A , analogously for V_n and B .

We have the following outline of A and $(U_n)_{n \geq o}$:



where $e_o < e_1 < e_2 < \dots$ are the indizes of these sets U_n which are infinite.

The same picture we have also for B and $(V_n)_{n \geq o}$ with the same indizes $e_o < e_1 < e_2 < \dots$.

The proof of this Theorem is a generalized version of the proof of Theorem 2.5.1 . Here $(U_n)_{n \geq o}$ and $(V_n)_{n \geq o}$ must be constructed simultaneous for ensuring 3^{oo} . But the construction principle is equal to that of 2.5.1 . The property of weak decreasing of $(U_n)_{n \geq o}$ and $(V_n)_{n \geq o}$ makes an additional step of the construction necessary. But , since we can claim for U_n and U_m with

$$n < m \,,\ U_n, U_m \text{ - infinite} \longrightarrow (U_n -_g U_m) \cap U_m = \emptyset \quad ,$$

we find infinitely many x from $U_n \cap A$ which we do not enumerate into U_m . We can do this and still ensure that $\overline{A} \subseteq U_m$ holds.

Let σ and τ be states of the same even length 2n. Let $\sigma \subset_0 \tau$ means that

$$\sigma(2i) = 1 \longleftrightarrow \tau(2i) = 1$$
$$\sigma(2i+1) = 1 \longleftrightarrow \tau(2i+1) = 1 \quad , \quad i=0,1,2,\ldots,n-1$$

and $\sigma \subset_1 \tau$ that

$$\sigma(2i) = 1 \longrightarrow \tau(2i) = 1$$
$$\sigma(2i+1) = 1 \longleftrightarrow \tau(2i+1) = 1 \quad , \ i=0,1,2,\ldots,n-1 \ .$$

We write $\sigma | \tau$ if

$$(\exists \rho)(\sigma \subset_0 \rho \wedge \tau \subset_1 \rho \ ) \qquad .$$

<u>Extension Theorem 3.3.3 (Soare)/So,74/</u>: Let A and B be infinite r.e. sets and $(U_n)_{n \geq 0}$, $(\tilde{V}_n)_{n \geq 0}$, $(\tilde{U}_n)_{n \geq 0}$, $(V_n)_{n \geq 0}$ recursive arrays and g a simultaneous enumeration of all these arrays and the sets A and B .

We assume the following two properties :

1) For all n

$$(3.6) \quad (A -_g \tilde{V}_n) \cap \tilde{V}_n = (B -_g \tilde{U}_n) \cap \tilde{U}_n = \emptyset \qquad .$$

For every even state σ with $|\sigma| = 2n$ we define subsets D^A_σ of A and D^B_σ of B .

Let D^A_σ be

$$\{x : (\exists s)(\, x \in A_{s+1} \setminus A_s \;\wedge\; (\,\sigma(2i)=1 \longleftrightarrow x \in U_{i,s}) \;\wedge\; (\,\sigma(2i+1)=1 \longleftrightarrow x \in \tilde{V}_{i,s})\,,\ i=0,1,\ldots,n-1)\}$$

and D^B_σ the set

$$\{x : (\exists s)(\, x \in B_{s+1} \setminus B_s \;\wedge\; (\,\sigma(2i)=1 \longleftrightarrow x \in \tilde{U}_{i,s}) \;\wedge\; (\,\sigma(2i+1)=1 \longleftrightarrow x \in V_{i,s})\,,\ i=0,1,\ldots,n-1)\}.$$

2) Suppose further that for every even state σ

$$(3.7)\quad \begin{array}{l} D^A_\sigma \text{ - infinite} \longrightarrow (\exists \tau)(\,\sigma \mid \tau \;\wedge D^B_\tau \text{ is infinite }) \\ D^B_\sigma \text{ - infinite} \longrightarrow (\exists \tau)(\,\tau \mid \sigma \;\wedge D^A_\tau \text{ is infinite }). \end{array}$$

Then there is a function p bijective between A and B and recursive arrays $(\hat{V}_n)_{n\geq 0}$ and $(\hat{U}_n)_{n\geq 0}$ with

$$\begin{array}{llll} \tilde{V}_n \subseteq \hat{V}_n & \text{and} & \tilde{V}_n \cap \bar{A} = \hat{V}_n \cap \bar{A} & \\ \tilde{U}_n \subseteq \hat{U}_n & \text{and} & \tilde{U}_n \cap \bar{B} = \hat{U}_n \cap \bar{B} & \qquad \text{for all } n \end{array}$$

and

$$(3.8)\quad \begin{array}{ll} p(A \cap U_n) =^* B \cap \hat{U}_n & \\ p^{-1}(B \cap V_n) =^* A \cap \hat{V}_n & \qquad \text{for all } n\ . \end{array}$$

Just the proof of this Theorem is long and technical complicated. An essential simplification of the original proof in /So,74/ was done by Maass in /Ma,83/ .

The basic idea is to construct two partial functions ψ_0 and ψ_1

$$\psi_0 : 2^{<\omega} \longrightarrow A \;, \quad \psi_1 : 2^{<\omega} \longrightarrow B$$

with

with $\psi_o(\sigma) := a$ if

$$st(a,|\sigma|) = \sigma ,$$

where $st(a,|\sigma|)$ is defined by $U_o, \hat{V}_o, U_1, \hat{V}_1, U_2, \ldots$, analogously for ψ_1 , B and $\hat{U}_o, V_o, \hat{U}_1, V_1, \hat{U}_2, \ldots$.
For these both functions it has to hold

(3.9) $\quad dom(\psi_o) \cap \mathcal{T}[\sigma]$ - infinite iff $dom(\psi_1) \cap \mathcal{T}[\sigma]$ is infinite

for all σ . The property (3.9) means just 3^o of the beginning of this subpoint.

By combining the both preceding Theorems we are able to prove that two maximal sets are automorphic. The first Theorem gives the assumption for the Extension Theorem and from this the automorphism follows.

Collorary 3.3.4(Soare): Two maximal sets are automorphic
Proof: Let A and B be maximal sets. First we construct for these sets recursive arrays $(U^o_n)_{n\geq o}$ and $(V^o_n)_{n\geq o}$ and an enumeration g_o such that for these all Theorem 3.3.2 holds .
We suppose that

$g_o(4s)$ enumerates A
$g_o(4s+1)$ enumerates $(U^o_n)_{n\geq o}$.
$g_o(4s+2)$ enumerates B
$g_o(4s+3)$ enumerates $(V^o_n)_{n\geq o}$.

Now let $(U_n)_{n\geq o}$ and $(V_n)_{n\geq o}$ be the arrays

$$U_n = \begin{cases} U^o_m & : \quad n = 2m \\ W_m \cap A & : \quad n = 2m+1 \end{cases} \qquad V_n = \begin{cases} V^o_m & : \quad n = 2m \\ W_m \cap B & : \quad n = 2m+1 \end{cases}$$

and $(\tilde{V}_n)_{n\geq o}$ and $(\tilde{U}_n)_{n\geq o}$ the arrays

$$\tilde{V}_n = \begin{cases} U^o_m & : \quad n = 2m \\ \emptyset & : \quad n = 2m+1 \end{cases} \qquad \tilde{U}_n = \begin{cases} V^o_m & : \quad n = 2m \\ \emptyset & : \quad n = 2m+1 \end{cases}$$

and g the simultaneous enumeration

$$\begin{aligned} g(6s) &= g_o(4s) \\ g(6s+1) &= g_o(4s+1) \\ g(6s+2) &- \text{ enumerates } W_n \cap A \\ g(6s+3) &= g_o(4s+2) \\ g(6s+4) &= g_o(4s+3) \\ g(6s+5) &- \text{ enumerates } W_n \cap B \end{aligned} .$$

Then A, $(U_n)_{n\geq o}$, $(\tilde{V}_n)_{n\geq o}$, B, $(\tilde{U}_n)_{n\geq o}$, $(V_n)_{n\geq o}$ and g satisfy the assumption of Theorem 3.3.3 (In (3.7) we even can take $\mathfrak{S}$ for $\mathfrak{T}$) .

Thus $(U_n)_{n\geq o}$, $(\hat{V}_n)_{n\geq o}$ and $(\hat{U}_n)_{n\geq o}$ and $(V_n)_{n\geq o}$ satisfy 2^o and 3^o . Since $(U_n)_{n\geq o}$ and $(V_n)_{n\geq o}$ generate $\mathfrak{E}^*$, we have $A^* \cong_{\mathfrak{E}^*} B^*$. From Theorem 3.1.4 we get $A \cong_{\mathfrak{E}} B$.

3.4 Corollaries and remarks to the automorphism proof .

From the automorphism result and the proof method of this in 3.3 it can be concluded still further properties of the maximal sets and other r.e. sets as we show in this point. Further we give here a survey of the Theorems which were proved until now by using this technique.

3.4.1 An equivalent formulation of Corollary 3.3.4

Exactly taken a stronger Theorem as the automorphism between two maximal sets was proved in 3.3 . We give here this stronger assertion and show to which this is equivalent .

Let X be a coinfinite r.e. set . With $\mathfrak{D}(X)$ we denote the family of sets

$$\{Y \text{ r.e. } : Y \subset^* X \vee X \cup Y = \omega\} .$$

In Corollary 3.3.4 it was shown that for two nonrecursive r.e. sets X_1 and X_2 the lattices $\mathcal{D}^*(X_1)$ and $\mathcal{D}^*(X_2)$ are isomorphic .
If X_1 and X_2 are maximal sets then $\mathcal{D}^*(X_1)$ as also $\mathcal{D}^*(X_2)$ are skeletons . Thus in this case the isomorphism (between $\mathcal{D}^*(X_1)$ and $\mathcal{D}^*(X_2)$) is already an automorphism of $\mathcal{E}^*$.

The isomorphism between $\mathcal{D}^*(X_1)$ and $\mathcal{D}^*(X_2)$ can be described still in an other way .
Let Z be an infinite set. Then Rec_Z means the unary relation

$$\{Y \text{ r.e. } : Y \text{ is recursive } \wedge Y \subseteq Z\} .$$

Lemma 3.4.1.1: Let X_1 , X_2 be nonrecursive r.e. sets . Then

$$(\mathcal{E} | X_1, \mathrm{Rec}_{X_1}) \cong (\mathcal{E} | X_2, \mathrm{Rec}_{X_2}) \text{ iff } \mathcal{D}^*(X_1) \cong \mathcal{D}^*(X_2).$$

This Lemma follows easily from the so-called Reduction principle which holds in $\mathcal{E}$. This principle formulated in the language of lattices with smallest element 0 says

$$(\forall x_1)(\forall x_2)(\exists y_1)(\exists y_2)(x_1 \vee x_2 = y_1 \vee y_2 \wedge y_1 \leq x_1 \wedge \wedge y_2 \leq x_2 \wedge y_1 \vee y_2 = 0) . \tag{3.10}$$

We ~~say~~ see that for r.e. sets X_1 and X_2 with $X_1 \cup X_2 = \omega$ the sets Y_1 and Y_2 for X_1 and X_2 in (3.10) are recursive. This is used in the proof of Lemma 3.4.1.1 .
But , since $\mathcal{D}^*(X_1)$ and $\mathcal{D}^*(X_2)$ are isomorphic , we get also the isomorphism between $(\mathcal{E} | X_1 , \mathrm{Rec}_{X_1})$ and $(\mathcal{E} | X_2 , \mathrm{Rec}_{X_2})$.
That $\mathcal{E} | X_1 \cong \mathcal{E} | X_2$ for all infinite r.e. sets is easy to see, see 3.1, Recursive permutations. From 3.3.4 it follows that there is an~~d~~ isomorphism between $\mathcal{E} | X_1$ and $\mathcal{E} | X_2$ preserving the recursive sets .

3.4.2 Further classes of r.e. sets and orbits in $\mathcal{E}$

By means of the isomorphism between $\mathcal{D}^*(X_1)$ and $\mathcal{D}^*(X_2)$ it can be concluded further automorphism and isomorphism properties in $\mathcal{E}$.

a) Quasimaximal sets

Corollary 3.4.2.1: Let A_1 and A_2 be q-maximal sets. Then

$$(3.11) \quad A_1 \cong_{\mathcal{E}} A_2 \text{ iff } \mathcal{E}^*(A_1) \cong \mathcal{E}^*(A_2) .$$

Thus the class of all q-maximal sets of the same order forms an orbit in $\mathcal{E}$.

Remark. The property (3.11) therefore that the isomorphism type of the structure of the r.e. supersets characterizes already the orbit probably is true only for the class $\mathcal{QM}$. For a big class of r.e. sets (3.11) is false as in /Ma,Sh,St/ and in /He,83/ it was shown.

b) $\mathcal{C}$ - maximal sets

A natural generalization of the notion of maximal set is the notion of $\mathcal{C}$ - maximal set.
Let $\mathcal{C}(X)$ be the family of all r.e. sets disjoint with X and $\mathcal{C}_0(X)$ be

$$\{X \cup Y : Y \in \mathcal{C}(X)\} .$$

Since $\mathcal{C}_0(X)$ is an ideal in $\mathcal{E}(X)$, there exists the factor structure $\mathcal{E}(X)/\mathcal{C}_0(X)$.

Definition 3.4.2.2: A coinfinite r.e. set X is called $\mathcal{C}$ - maximal if $\mathcal{E}(X)/\mathcal{C}_0(X)$ has only two elements.

Equivalent to this is the requirement that for every $Y \in \mathcal{E}(X)$

$$Y \setminus X \in \mathcal{E}(X) \quad \text{or} \quad (\exists Z)(Z \in \mathcal{E}(X) \wedge Y \cup Z = \omega) \quad .$$

We see easily that a set X is maximal iff X is simple and $\mathcal{E}$ - maximal .

By using 3.4.1.1 the automorphism relation between two $\mathcal{E}$ - maximal sets can be reduced to the structure of $\mathcal{E}(\cdot)$.

Lemma 3.4.2.3: Let A_1 and A_2 be two $\mathcal{E}$ - maximal sets. Then

$$(\mathcal{E}(A_1), Rec_{\overline{A_1}}) \cong (\mathcal{E}(A_2), Rec_{\overline{A_2}}) \quad \text{iff} \quad A_1 \cong_{\mathcal{E}} A_2 \quad .$$

In the following we give six classes of $\mathcal{E}$ -maximal sets. By means of 3.4.2.3 it can be shown that these classes are all orbits in $\mathcal{E}$.

1. The r.e. sets X with $\mathcal{E}(X)$ consisting only of finite sets (the maximal sets)
2. $\mathcal{E}(X)$ has a greatest element Y (mod Fin) which is an infinite recursive set (X is maximal in the recursive set Y) .
3. $\mathcal{E}(X)$ has a greatest element Y (mod Fin) which is not recursive (X and Y are a splitting of the maximal set $X \cup Y$)

We denote with $\oplus \mathcal{E}$ the weak product of $\mathcal{E}$. This is the lattice with the basic set

$$\{ f : f : \omega \longrightarrow \mathcal{E}, \ (\forall a.a.n)(f(n) = \emptyset) \} \quad .$$

with the operations : $f \wedge g$ and $f \vee g$ defined by

$$(f \wedge g)(i) = f(i) \cap g(i) \quad , \quad (f \vee g)(i) = f(i) \cup g(i) \quad , \ i \geq 0 .$$

We say that $\mathcal{E}(X) \cong \oplus \mathcal{E}$ respectively to the array of r.e. sets $(X_n)_{n \geq 0}$ if $X_n \in \mathcal{E}(X)$, $(X_n)_{n \geq 0}$ is disjoint and

$$\varphi(X) = \{\bigcup_{i=0}^{\infty}\{g_i(f(i)) : f \in \oplus\mathcal{E}\}\} ,$$

where g_i are fixed enumerations of X_i , $i \geq 0$.

4. $\varphi(X) \cong \oplus\mathcal{E}$ resp. to an array of recursive sets
5. $\varphi(X) \cong \oplus\mathcal{E}$ resp. to an array $(X_n)_{n\geq 0}$, where all X_n except X_0 , are recursive (X and X_0 are splittings of $X \cup X_0$ which belongs to 4.) .
6. $\varphi(X) \cong \oplus\mathcal{E}$ resp. to an array of nonrecursive r.e. sets .

The last three classes are not empty , see /He,81/ .

2 - orbits of $\mathcal{E}$

From the automorphism construction in subpoint 3.3 we get also 2-orbits of $\mathcal{E}$. A class $\mathfrak{X}$ of $\mathcal{E} \times \mathcal{E}$ is a 2-orbit if $\mathfrak{X}$ is not empty and

$$(\forall (X_1,Y_1) \in \mathfrak{X})(\forall (X_2,Y_2) \in \mathfrak{X})(\exists \Phi \in \mathrm{Aut}(\mathcal{E}))$$
$$(\Phi(X_1) = X_2 \wedge \Phi(Y_1) = Y_2) .$$

If $\mathfrak{X}$ is a 2-orbit then of course there are orbits $\mathcal{O}_1$ and $\mathcal{O}_2$, s.t. $\mathfrak{X} \subseteq \mathcal{O}_1 \times \mathcal{O}_2$ and every product of two orbits $\mathcal{O}_1 \times \mathcal{O}_2$ is an union of 2-orbits .

We have the following 2-orbits in $\mathcal{E}$:

1. $\{(M_1,M_2) :$ M_1 - maximal M_2 - maximal , $M_1 \neq^* M_2\}$
2. $\{(R,M) :$ R - recursive, infinite , M - maximal, $R \subseteq M\}$
3. $\{(A_1,A_2) :$ (A_1,A_2) is a Friedberg splitting of $A_1 \cup A_2$ which is maximal $\}$

c) Hyperhypersimple sets

Between the classes $\mathcal{QM}$ and $\mathcal{DS}$ there is still a further important class of simple sets.

Definition 3.4.2.4(Post): An infinite set X is called hyperhyperimmune (short: hh-immune) if there is no recursive array of disjoint sets $(U_n)_{n \geq 0}$, s.t.

$$(\forall n)(\ U_n \text{ is finite}) \wedge (\forall n)(\ U_n \cap X \neq \emptyset\ ) \quad .$$

A set X is called hyperhypersimple (hh-simple) if X is r.e. coinfinite and $\overline{X}$ is hh-immune.

In /La,68a/ it was shown that a coinfinite r.e. set X is hh-simple iff $\mathcal{E}^*(X)$ is a Boolean algebra . Thus we have $\mathcal{QM} \subsetneq \mathcal{HH}$(the class of hh-simple sets) . The inclusion $\mathcal{HH} \subsetneq \mathcal{DS}$ was proved in /Ma,63/.

For the hh-simple sets the characterization of automorphic hh-simple sets is much more complicated as for q-maximal sets. In /He,83/ it was shown that for every infinite Σ^0_3-Boolean algebra $\mathcal{L}$ (for the definition of Σ^0_3 Boolean algebra , see /So,78/) the class

$$\{\,X \text{ is hh-simple} :\ \mathcal{E}^*(X) \cong \mathcal{L}\,\}$$

decomposes into infinitely many orbits .

A sufficient condition that two hh-simple sets are automorphic will be given in 3.4.3 f) .

Semirecursive sets

A set X is called semirecursive if there is a recursive function f of two variables , s.t. for all x and y

$$f(x,y) = x \quad \text{or} \quad f(x,y) = y$$
$$x \in A \ \vee\ y \in A \longrightarrow f(x,y) \in A \quad .$$

There are very many nonrecursive r.e. sets which are semirecursive , even inside the class of h-simple sets as Jockusch showed , see /Od,p.51/ . But no hh-simple set is semirecursive, Martin /Od,p.51/. Thus

Corollary 3.4.2.5: Maximal sets are not semirecursive .

d) Major subsets.

In /La,68a/ was introduced the binary relation (in $\mathcal{E}$) " X is a major subset of Y " .

Definition 3.4.2.6: Let A and B be r.e. sets with $A \subseteq B$. A is called major subset of B if $A \subset_\infty B$ and

$$(3.12) \qquad (\forall C \text{ r.e.})(B \cup C = \omega \longrightarrow A \cup C =^* \omega) \qquad .$$

Every nonrecursive r.e. set has such special subsets, see /La,68a/. Let X be a nonrecursive r.e. set and $\mathfrak{M}\mathcal{S}(X)$ the class of all major subsets of X .
Since (3.12) is equivalent to

$$(\forall R \text{ rec.})(R \subseteq B \longrightarrow R \subset^* A) \qquad ,$$

by the Reduction principle , see 3.4.1 , it follows that all lattices $\mathfrak{M}\mathcal{S}(X)$ are isomorphic between each other ~~all~~ for all nonrecursive r.e. sets X .

An other isomorphism result for major subsets is given in 3.4.3 e) .

3.4.3 Automorphism and isomorphism results proved with Theorem 3.3.3

The technique of the automorphism construction for maximal sets, therefore the construction of special recursive arrays to which the Extension Theorem is applicable , was also used for proofs by using this method :

a) In /St,82a/ it was shown that the class of dense simple sets is not lattice definable .

b) In /So,82/ an isomorphism property was proved which in c) was generalized yet.

c) In /Ma,83/ there was proved an Theorem which is given in 8.2.6 .

d) In /Sch/ this method was used for proving a fact which is mentioned in 9.3 .

e) In /Ma,St/ it was shown that all lattices $\mathcal{E}^*(A,B)$ with A major subset of B are isomorphic between each other.

f) In /Ma,84/ the following sufficient condition for the automorphism of two hh-simple sets was shown :

Let A and B be hh-simple sets. If there is a Δ^0_3 permutation p , s.t.

$$(A \cup W_e)^* \in \mathcal{E}^*(A) \longrightarrow (B \cup W_{p(e)})^* \in \mathcal{E}^*(B)$$

is an isomorphism , then A and B are automorphic .

3.4.4 Inducing and presenting permutations for the automorphisms between maximal sets

Since two maximal sets are automorphic , there are permutations which induce and present this automorphism. We will here give the estimation of these permutations inside the arithmetical hierarchy .

Lemma 3.4.4.1: For two maximal sets there is a permutation from Δ^0_2 which induces an automorphism between both.

Lemma 3.4.4.2: For two maximal sets there is a permutation from Δ^0_3 which presents an automorphism between both .

The first Lemma uses the fact that the mapping

$$U_n \longrightarrow \hat{U}_n \quad \text{and} \quad \hat{V}_n \longrightarrow V_n$$

is an automorphism and all arrays are recursive .

The second one uses the fact that the array $(U_n)_{n\geq 0}$ from 2.5.1 is a skeleton and there is a function f from Σ^0_3 with $W_n = U_{f(n)}$. Further $U_n = W_{h(n)}$ for some recursive function h . Thus we have

$$W_e = U_{f(e)} \longrightarrow \hat{U}_{f(e)} =^* W_{h(f(e))}$$

and $h \circ f \in \Sigma^0_3$. Analogeously for the converse mapping .

3.4.5 The group Aut($\mathcal{E}^*$) and the maximal sets

Let Ω be a subclass of $\mathcal{E}$. We say that Ω generates Aut($\mathcal{E}^*$) if

$$(\forall \Phi^* \in \mathrm{Aut}(\mathcal{E}^*))((\forall X \in \Omega)(\Phi^*(X^*) = X^* \longrightarrow \Phi^* = \mathrm{Id}^*)) .$$

Lemma 3.4.5.1: If Ω includes an orbit of an infinite and coinfinite r.e. set respectively to $\mathrm{Aut}_r(\mathcal{E}^*)$, then Ω generates $\mathrm{Aut}(\mathcal{E}^*)$.

Corollary 3.4.5.2: The class Max generates $\mathrm{Aut}(\mathcal{E}^*)$.

Thus , for charaterizing an automorphism it is sufficient to know how he transform the maximal sets .

Let M_1^* , M_2^* ,... be a sequence of all maximal sets (resp. to $=^*$) without repetition . Then the assignment :

$$\Phi^* \in \mathrm{Aut}(\mathcal{E}^*) \longrightarrow P_{\Phi} \in S_\omega \quad (S_\omega \text{ - the permutation group of } \omega)$$

with

$$P_{\Phi}(i) = j \quad \text{iff} \quad \Phi^*(M_i^*) = M_j^*$$

is an embedding from $\mathrm{Aut}(\mathcal{E}^*)$ into S_ω , by 3.4.5.2 .

From 3.3.4 we know

$$(\forall i)(\forall j)(\exists P_{\Phi})(P_{\Phi}(i) = j \wedge P_{\Phi}(j) = i) .$$

Since for every $P_{\Phi} \neq \mathrm{Id}$ the set $\{ i : P_{\Phi}(i) \neq i \}$ is infinite , $\mathrm{Aut}(\mathcal{E}^*)$ is not isomorphic with S_ω .

3.4.6 Automorphisms and sequences of maximal sets

If $(M_1, M_2, \ldots, M_n)$ and $(M_1', M_2', \ldots, M_n')$ are sequences of pairwise not equal maximal sets (mod Fin) , then by 3.4.2.1 there is an automorphism Φ^* with $\Phi^*(M_i^*) = M_i'^*$, $i=1,\ldots,n$.

How is the situation for infinite sequences of maximal sets ? For arbitrary , infinite sequences of pairwise not equal maximal sets (mod Fin) in general such an automorphism does not exist (One ~~sequences~~ sequence can ~~consists~~ consist of maximal sets including all an infinite r.e. set (mod Fin) , but the second one is not so) .

Exists an automorphism if the sequences are recursive arrays ?

REDUCIBILITY THEORY

One of the most important basic notion in the recursion theory is the notion of the reducibility .

A reducibility is a comparison of two objects (e.g. sets of functions) respectively to the complexity of their calculation. We say that the set X is reducible to a set Y respectively to a regarded measure of effectivity if X has a ~~higher~~ lower level of effectivity as Y respectively to the given measure.

We give here a precise definition of a reducibility and briefly a general introduction in this topic . In the points 4.5 and 6. there are given detailed results of special reducibilities and their connections to the maximal sets .

Basic notions

A binary relation $\leq_r$ in $\mathcal{P}(\omega)$ is called <u>(abstract) reducibility</u> if these relation is reflexive (i.e. $X \leq_r X$ for all $X \in \mathcal{P}(\omega)$) and transitive (i.e. $X \leq_r Y \wedge Y \leq_r Z \,.\!\longrightarrow X \leq_r Z$, $X,Y,Z \in \mathcal{P}(\omega)$) .

A reducibility $\leq_r$ is called <u>effective</u> if from $X \leq_r Y$ it follows that there is a(n) (effective) algorithm by means of it and the informations if $n \in Y$ or not for an arbitrary n the set X can be calculated (i.e. that for every number x it ~~e~~ can be decided if $x \in X$ or not) .

(A precise mathematical definition of the notion "effective reducibility " will be given in point 4.) .

A reducibility $\leq_r$ is called <u>arithmetical</u> (or <u>describeable</u>) if $\leq_r$ has an arithmetical definition in $\mathfrak{N}^2$ ($\mathfrak{N}^2 = (\omega, +, \cdot, \leq, {}', 0, \mathcal{P}(\omega), \in)$ - the standard model of the second order arithmetic) , i.e. there is a fomula φ having only quantifiers over numbers , s.t.

$$X \leq_r Y \longleftrightarrow \mathfrak{N}^2 \models \varphi[X,Y] \qquad \text{for all } X,Y \in \mathcal{P}(\omega).$$

If $\leq_r$ is a reducibility then the relation $X \equiv_r Y$ which is defined by $X \leq_r Y \wedge Y \leq_r X$ is an equivalence relation in $\mathcal{P}(\omega)$ and the factor structure $(\mathcal{P}(\omega)_{/\equiv_r}, \leq_{r/\equiv_r})$ is a partial ordering . We denote this ordering with $\mathcal{Q}_r$.
The equivalence classes are called r-degrees . For the r-degree to which the set $X \in \mathcal{P}(\omega)$ belongs we write $[X]_r$. If $\mathcal{X}$ is a subclass of $\mathcal{P}(\omega)$ then $dg_r(\mathcal{X})$ denotes the degree class $\{[X]_r : X \in \mathcal{X}\}$. With 0 we denote the smallest r-degree . That is the r-degree consisting of all recursive sets.

Reducibilities and recursively enumerable sets

We regard in this paper the reducibilities not in general but only restricted to the r.e. sets (and sometimes also to the Δ^0_2 sets) .

If $\leq_r$ is a reducibility and d a r-degree then d is called r.e. degree (d is an r.e. r-degree) if d includes an r.e. set. With $\mathcal{Q}_{r,r.e}$ we denote the restriction of $\mathcal{Q}_r$ to the class of r.e. r-degrees and with $\mathcal{Q}_{r,\Delta^0_2}$ the subordering of $\mathcal{Q}_r$ consisting of all r-degrees with Δ^0_2 sets .
Since recursive in Δ^0_2 remains Δ^0_2 , for effective reducibilities $\mathcal{Q}_{r,\Delta^0_2}$ is an initial part of $\mathcal{Q}_r$. But $\mathcal{Q}_{r,r.e.}$ is only a subordering of $\mathcal{Q}_{r,\Delta^0_2}$ which in general essentially differs from $\mathcal{Q}_{r,\Delta^0_2}$.
A certain r.e. set plays an important role in the analyse of $\mathcal{Q}_{r,r.e.}$ and $\mathcal{Q}_{r,\Delta^0_2}$. The r.e. set $\{e : W_e(e)\}$ is called creative set and is denoted with K .
The importance of K follows from the property that every r.e. set is reducible to K respectively to all reducibilities treated in this paper .

Let $\leq_r$ be a reducibility . A set X is called r-complete if X is r.e. and every r.e. set Y is r-reducible to X .
With $0'_r$ we denote the r-degree of r-complete sets. and call it the complete r-degree. From above it follows that $K \in 0'_r$ for all $\leq_r$ and thus the r-degree $0'_r$ exists (is not empty) .
In general $0'_r$ includes also Δ^0_2 sets which are not r.e. and for different reducibilities the classes of their complete sets

do not coincide. O'_r is the greatest element in $\mathcal{Q}_{r,r.e.}$ (but not in $\mathcal{Q}_{r,\Delta_2^0}$ for all $\leqslant_r$).

Reducibilities and maximal sets

In the following points the connections between the reducibilities and the maximal sets are investigated.
This topic can be divided into four points of view.

1° Let given a(n) (effective) reducibility $\leqslant_r$.
- What can said about the partial ordering of r-degrees with maximal sets ?
- What is the position of the r-degrees with maximal sets inside $\mathcal{Q}_{r,r.e.}$ and $\mathcal{Q}_{r,\Delta_2^0}$?

2° The characterization of the r.e. sets r-equivalent to a maximal set (i.e. $\{X \text{ r.e. } : X \equiv_r M\}$, M - maximal) . At this in particulary the maximal sets M_1 , M_2 with $M_1 \equiv_r M_2$.

3° For the maximal sets M_1 amd M_2 with $M_1 \equiv_r M_2$ to characterize the reduction algorithms for $M_1 \leq_r M_2$ and $M_1 \leqslant_r M_2$.

4° The comparision of two reducibilities and the role of the maximal sets at this .

Let $\leqslant_{r_1}$ and $\leqslant_{r_2}$ be reducibilities . We say that $\leqslant_{r_2}$ *is stronger than* $\leqslant_{r_1}$ if

$$(\forall X)(\forall Y)(X \leqslant_{r_1} Y \longleftrightarrow X \leqslant_{r_2} Y) \quad .$$

Suppose $\leqslant_{r_2}$ is a stronger reducibility as $\leqslant_{r_1}$.
- What can said about $\leqslant_{r_1}$ and $\leqslant_{r_2}$ inside the class Max ?
- Let d be a r_2-degree of a maximal set. Then inside the class of r.e. sets in d the same analyse as in 1° , 2° and 3° can be provided .
- What can said about the r_1-degrees of maximal sets inside $\mathcal{Q}_{r,r.e.}$ when all maximal sets are r_2-equivalent ?

4. TURING DEGREES OF MAXIMAL SETS , TURING - COMPLETENESS

We begin the investigations of the relationship between the reducibilities and the maximal sets with the Turing reducibility. This reducibility is the most general (effective) reducibility, since for calculating the set X any algorithm can be used. Few Theorems about the maximal sets and their relationship to the T-reducibility were shown. But we shall see that the T-reducibility is complicated not only inside all r.e. sets but also when we restrict this to the class of maximal sets. At the end of this point the T-complete maximal sets are still separately investigated.

4.1 Turing - reducibility and jump classes . With $\{\varphi_e^A\}_{e \geq o}$ we denote the standard f-array of all partial functions recursive in A . This f-array has the representation

$$\varphi_e^A(x) \simeq y \longleftrightarrow (\exists z)R(\hat{\sigma}(A,z),e,x,y) ,$$

where R is some recursive relation , see /Se,67/,p.171 (and $\hat{\sigma}(A,z) = \hat{\sigma} \wedge \sigma = \sigma(A,z)$, see the Introduction) .

Definition 4.1.1: We say that X is Turing- reducible to Y (shortly, X is T-reducible to Y , $X \leq_T Y$) if C_X is equal to φ_e^Y for some e .

With $\{W_e^A\}_{e \geq o}$ we denote the standard array of all sets recursively enumerable in A , hence $W_e^A = \{x : \varphi_e^A(x)\downarrow\}$.

General properties of the T-reducibility and in particulary of the r.e. T-degrees can be found e.g. in /So,83/ .

The investigations of the T-degree of the maximal sets were begin with the proofs that there are as well as T-complete maximal sets . After this Martin gaves a complete characterization of the T-degrees of the maximal sets.

We call a simple set S effectively simple if

$$(\exists\, g \text{ rec. fct})(\forall e)(\, W_e \subseteq \bar{S} \longrightarrow |W_e| \leqslant g(e)\,) \quad .$$

Effectively simple sets are T-complete as in /Mat,66a/ it is shown. We show this later in subpoint 4.5 .

If we replace in Theorem 1.1.2 the requirement (1.1) by

$$(\exists n)(\ (\exists e)(\ e<n \wedge m^s(n) \in W_{e,s} \wedge W_{e,s} \cap M_s = \emptyset\ ) \vee$$
$$\vee (\exists l)(n<l \wedge W(m^s(n),n,s) < W(m^s(l),n,s))\ ) \quad .$$

Then we get an effectively simple maximal set, /Ya,65/ , g is in this case the function $g(e) = e+1$, and thus a T-complete maximal set.

By a suitable combination of the Friedberg-Muchnik technique, see e.g. /Sh,67/ p. 170 and the Friedberg's maximal set construction we can construct two maximal sets with incomparable T-degrees , see /Sa,64/ . These both maximal sets are of course not T-complete .

The complete characterization of the T-degrees of the maximal sets will be done in subpoint 4.2 .

Turing - degrees and jump classes

Denote with A' (or also K^A) the jump of A (i.e. the set $\{x : x \in W_x^A\}$ and with d' the jump of the degree d . For

details see /Ro,§13.1/.
With $d^{(n+1)}$ we denote the degree $(d^{(n)})'$ ($d^{(o)} = d$) .
Thus e.g. $0^{(o)} = 0$, $0^{(1)} = 0'$, ... ,
Let

$$\mathbb{H}^n = \{ d : d \leq 0' \wedge d^{(n)} = 0^{(n+1)} \}$$

and

$$\mathbb{L}_n = \{ d : d \leq 0' \wedge d^{(n)} = 0^{(n)} \} .$$

The degrees in $\mathbb{H}^n$ are called <u>n-high</u> and the degrees in $\mathbb{L}_n$ <u>n-low</u> . For 1-high and 1-low we say shortly high and low respec~~tivl~~tively.

From the definitions it follows

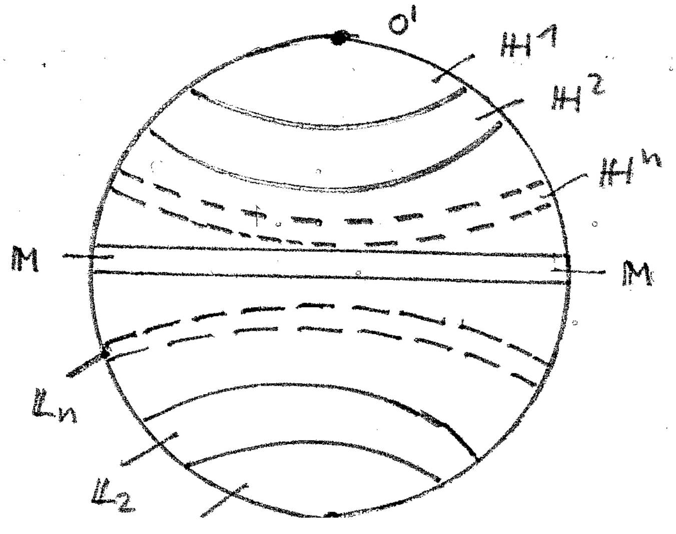
$$(4.1) \qquad \mathbb{H}^o \subseteq \mathbb{H}^1 \subseteq \mathbb{H}^2 \subseteq \ldots , \quad \mathbb{L}_o \subseteq \mathbb{L}_1 \subseteq \mathbb{L}_2 \subseteq \ldots$$

and that all classes $\mathbb{H}^n$ and $\mathbb{L}_m$ are disjoint for all n and m. It is well-known that all inclusions in (4.1) are proper even if all classes are restricted to the r.e. T-degrees. Further it is known that the class $\mathbb{M}$ equal to

$$\{ d : d \leq 0' \wedge d \notin \bigcup_{n \geq 0} \mathbb{H}^n \wedge d \notin \bigcup_{n \geq 0} \mathbb{L}_n \}$$

is not empty , more includes even r.e. T-degrees , see /So,78/.

Thus we have the following scheme about the T-degrees below 0' by meaning of the jump operator :



In the following we say shortly n-high and n-low for T-degrees which are n-high and r.e. and n-low and r.e. resp. , since in this paper we are interested only in these .

The Δ_2^0 sets defined in the Introduction are nearly connected with the T-reducibility .
It holds for sets X and Y

$$X \text{ is } \Delta_2^0 \text{ in } Y \quad \text{iff} \quad X \leq_T Y' \qquad ,$$

see e.g. /Ro,p.403/.
Thus in particulary a set X is Δ_2^0 iff X is T-reducible to K . This , not difficult to show , basic fact is oftenly used in the Recursion theory .

<u>4.2 Turing - degrees of maximal sets</u> . By using the jump operator , more precisely the jump class $\mathbb{H}^1$ the T-degrees of the maximal sets can be characterized. As we shall see the notion of dominating functions introduced in point 2.2.2 plays here the fundamental role. This connection between the T-reducibility and the dense simple sets was shown by Martin in /Mat,66b/ .

<u>Theorem 4.2.1 (Martin)</u>: Let d be a T-degree . Then

$$0'' \leq d' \quad \text{iff} \quad (\exists f)\ fct)(\ f \text{ is recursive in } d \wedge \wedge\ f \text{ is dominant }) \quad .$$

<u>Corollary 4.2.2</u>: The T-degree of a dense simple set is high
<u>Proof</u>: If X is dense simple then $p_{\bar{X}}$ is dominant, see 2.2.2. Further $X \equiv_T p_{\bar{X}}$ for every set X . Thus $p_{\bar{X}}$ is recursive in $[X]_T$ and dominant . By Theorem 4.2.1 we get $0'' \leq_T [X]_T'$. Since X is r.e. , $[X]_T' \leq_T 0''$. Both together gives $0'' = [X]_T'$. Hence $[X]_T$ is high .

Corollary 4.2.3: The T-degrees of the maximal sets are high.

The stronger notion " strongly dominant " also implies an special T-degree property which we give in subpoint 4.5 .

Knowing that all T-degrees of the maximal sets are high it arises the question if all high degrees include maximal sets ?

Theorem 4.2.4 (Martin): Every high T-degree includes maximal sets

Corollary 4.2.5 (Martin): A T-degree includes a maximal set iff this degree is high .

Remark. The T-degrees of h-simple sets include all r.e. T-degrees, except 0 , Dekker . But the h-simple sets constructed there are all co-retraceable , see /Ro,p.206/ and thus not fs $\mathcal{H}\mathcal{S}$, by 2.4.1 , 1) . In /So,83/ it is shown that all sets of fs $\mathcal{H}\mathcal{S}$ also have high T-degrees.

4.3 Maximal sets with special degree properties . From Corollary 4.2.5 it follows that the T-reducibility inside the class of degrees with maximal sets is the same as inside the class of degrees with dense simple sets and this is $(H^1, \leq_T)$ is very complicated and an extensive topic of investigations for itself .
The high degrees were already analysed in several papers. Here we mention only few results about it which were shown by constructing special maximal sets .

Minimal pairs of r.e. T-degrees

We say that two degrees a and b form a minimal pair if a ≠ 0 and $b \neq 0$ but $a \wedge b = 0$ (a and b are bounded downwards only by 0) .

Theorem 4.3.1 (Lachlan)/La,66b/: There are two maximal sets , s.t. their T-degrees form a minimal pair.

Thus there are even high degrees which form a minimal pair , throughout there is no pair of r.e. T-degrees a , b with $a \wedge b = 0$ and $a \vee b = 0'$, see /So,78/ . Theorem 4.3.1 also is of interest from the view point of the construction method of these maximal sets , see /So,t.a./ .

Mitotic r.e. sets , splitting theorem

An other topic in the theory of the r.e. sets is the analysis of the possible T-degrees of the splitting sets of an r.e. set. If B and C are splitting sets of A then of course $[B]_T \cup [C]_T = [A]_T$. What can said more about the degrees of the splitting sets ?

An r.e. set A is called mitotic if there is a splitting B and C of A , s.t. both sets B and C have the same Turing degree as A . A is called nonmitotic otherwise .

Theorem 4.3.2 (Ladner)/Lad/: There exist maximal sets which are mitotic and maximal sets which are nonmitotic .

Closely connected with this is the following notion :
An r.e. set A has the universal splitting property (USP) if for any r.e. set D with $D \leq_T A$ there is a splitting B and C of A , s.t. B and D are Turing equivalent ($B \equiv_T D$) , see /Le,Re/. Still open is the analyse of the maximal sets under this point of view . Theorem 4.3.2 gives a first answer inside this area.

Incomparability of T-degrees

Marczenkov proved a Theorem about maximal sets which is such general that it holds for all reducibilities regarded in this paper and thus in particular also for the T-reducibility.

Theorem 4.3.3 (Marczenkov)/Mar/: If d is a r.e. T-degree with $d \neq 0$ and $d \neq 0'$, then there is a maximal set M s.t. d and $[M]_T$ are incomparable .

In the next points 5 and 6 in corresponding places we shall refer this Theorem and that what is said before it.

4.4 Promtly simple sets , high degree splitting .

In /Ma,82/ Maass introduced a new notion which was deviced similar as the notion of effective simple set directly from Post's simple set construction. This notion is the third after the investigations in 2.5 and 2.6 which produces a connection between r.e. sets and enumerations of r.e. sets. Above all for the analyse of r.e. T-degrees this new notion showed up usefull.

Let S be a simple set and W_e infinite . Let g be a simultaneous enumeration of $\{W_e\}_{e \geq 0}$ and h an enumeration of S . It will be analysed the connection between s and t for x with $x \in W_{e,s} \setminus W_{e,s-1}$ and $x \in S_t$.

Definition 4.4.1 (Maass): A coinfinite r.e. set A is called promtly simple (resp. to the given enumeration of $\{W_e\}_{e \geq 0}$) if there is a nondecreasing recursive function p and an enumeration of A ,s.t. for every $e \geq 0$

$$(4.2) \qquad W_e \text{ is infinite} \longrightarrow (\exists s)(\exists x)(x \in (W_{e,s} \setminus W_{e,s-1}) \cap \\ \cap A_{p(s)}) .$$

Thus for promtly simple sets there exists a recursive function p , s.t. for every infinite W_e at least for one $x \in W_e \cap A$ x is enumerated into A not later than after p(s) steps, assuming that x cames to W_e in the enumeration step s .

Promtly simple sets are simple by definition. The Post's simple set is promtly simple , since $p(s) = s+1$ satisfies (4.2). The following Theorem shows that the notion of promtly simple is independent from the enumeration of $\{W_e\}_{e \geq 0}$:

Theorem 4.4.2 /Ma,Sh,St/: A coinfinite r.e. set A is promtly simple iff there is a recursive function f , s.t. for all e the following three properties hold

$$W_{f(e)} \subseteq W_e$$
$$W_{f(e)} \cap \bar{A} = W_e \cap \bar{A}$$
$$W_e \text{ is infinite} \longrightarrow W_e \setminus W_{f(e)} \cancel{\neq \emptyset}\ ^{\neq \emptyset} .$$

Denote with $\mathcal{P}r\mathcal{S}$ the class of all promtly simple sets .

Remark 4.4.3 . The class $\mathcal{P}r\mathcal{S}^o$ forms a filter in $\mathcal{E}^*$, properly included in $\mathcal{S}^o$ and not lattice definable . The class $dg_T(\mathcal{P}r\mathcal{S})$ forms also a filter in $\mathcal{A}_{T,r.e.}$,not including all high degrees . See for these all /Ma,Sh,St/ .

Above all we are here interested in the relationship between Max and $\mathcal{P}r\mathcal{S}$. It holds

Theorem 4.4.4 (Maass,Shore,Stob): There are maximal sets which are promtly simple and maximal sets which are not promtly simple.

The existence of maximal sets which are also promtly simple is easy to show . Provide for all odd construction steps take an m_t^s , $t > e$ to M_{s+1} if $m_t^s \in W_{e,s}$ but $M_s \cap W_{e,s} = \emptyset$. If e is chosen respectively to the minimal s with $m_t^s \in W_{e,s}$ and $M_s \cap W_{e,s} = \emptyset$ we get a minimal promtly simple set with $p(s) = s+1$. An other argumentation is given in /Ma,Sh,St/ .
Since not all high degrees belong to $dg_T(\mathcal{P}r\mathcal{S})$, see Remark 4.4.3

from Theorem 4.2.4 it follows that there are maximal sets which are not promtly simple.

High degree splitting

The question if there is a lattice - definable class of r.e. sets $\mathcal{X}$, s.t. $dg_T(\mathcal{X})$ splits $\mathbb{H}^1$ cannot be answered by meaning of $\mathcal{P}+\mathcal{S}$, since this class is no lattice definable class. It will be shown here that there is a greater class of r.e. sets lattice definable and splitting $\mathbb{H}^1$.

We get a class by generalizing the Friedberg splitting of an r.e. set .

Definition 4.4.5 (Maass,Shore,Stob): We say that a r.e. set A has the splitting property if every nonrecursive r.e. set B has a Friedberg splitting (B_0,B_1) with $B_0 \subseteq A$.

Let Sp be the symbol for the family of sets with the splitting property. Friedberg's splitting Theorem says that all cofinite sets belong to Sp . From the definition it follows that Sp is a subfamily of $\mathcal{S}$ and that Sp is a lattice definable class of r.e. sets .

The class Sp includes all promtly simple sets and all hh-simple sets and forms a filter in $\mathcal{E}$. For this and others , see /Ma,Sh,St/ .

Theorem 4.4.6 /Ma,Sh,St/: The family Sp $\setminus \mathcal{H}\mathcal{H}$ nontrivially splits all classes $\mathbb{H}^n$ and $\mathbb{L}_n$ for every $n \geq 1$.

Remark 4.4.7. Theorem 4.4.6 refutes the conjecture of Soare and gives an answer of the question of Friedman (Problem No. 60 , /Fr/) if $dg_T(\mathcal{O}_{\mathcal{E}}(X))$ for X nonrecursive r.e. set always includes $\mathbb{H}^1$. Nevertheless it is to conjecture that $dg_T(\mathcal{O}_{\mathcal{E}}(X))$ is closed upwards , for every X as above .

4.5 Criterions for Turing - completeness of maximal sets . In this subpoint the maximal sets which are T-complete will be analysed . At this we are interested in properties and criterions which ensure T-completeness. It will be shown that there are very different properties which implies T-completeness.

For the investigation of T-completeness of an arbitrary r.e. set there is a criterion with a fundamental meaning. Just by using this other criterions can be proved quite easily . The following very clearly criterion is the final version of Theorems to this problem , see /So,78/ :

Theorem 4.5.1 (Arslanov)/So,82/,/Od/: Let A be a r.e. set . A is T-complete iff

$$(\exists f \text{ fct}, \text{ recursive in } A)(\forall e)(W_e \neq W_{f(e)})$$

(the function f has no fix point) .

a) Already in point 2.2.2 the strongly dense simple sets were defined. The following Lemma shows a property of these sets :

Lemma 4.5.2 (Tennenbaum)/So,78/: Let f be a strongly dominant function. Then $K \leq_T f$.
Proof: Let $\varphi_k(x) = (\mu s)(x \in K_s)$. Since $\varphi_k(x) \leq f(x)$ for almost all x from K we have $x \in K$ iff $x \in K_{f(x)}$ (mod Fin) . Hence $K \leq_T f$.

Corollary 4.5.3: Every strongly dense simple set is T-complete .

It is easy to construct a set which is maximal and simultaneous strongly dense simple . Thus we have an easy method for constructing T-complete maximal sets.

b) The notion of effectively simple set defined in the beginning of this point and modifications of this are meaningfull for this topic.

A simple set S is called <u>weak effective simple</u> , if there is a function g recursive in S , s.t.

$$(\forall e)(W_e \subseteq \bar{S} \longrightarrow |W_e| \leqslant g(e)) \quad .$$

The following Lemma shows the connection between weak effective simple and T-completeness :

<u>Lemma 4.5.4 /So,82/</u>: Let S be a simple set. S is T-complete iff S is weak effectively simple .

Thus weak effective simple characterizes completely the T-complete simple sets .

<u>Corollary 4.5.5 (Martin)/Mat,66a/</u>: Effectively simple sets are T - complete .

Thus the maximal set mentioned in the subpoint 4.1 is T-complete.

There is also an itensification of the notion of effectively simple set .

A simple set S is called <u>strongly effectively simple</u> if there is a recursive function g , s.t.

$$(\forall e)(W_e \subseteq \bar{S} \longrightarrow \max |W_e| \leqslant g(e)) \quad .$$

<u>Theorem 4.5.6 (Cohen,Jockusch)/Co,Jo/</u>: Strongly effective simple sets are not dense simple .

<u>Corollary 4.5.7</u>: Maximal sets are not strongly effectively simple sets .

Thus there are maximal sets which are effectively simple , but not strongly effectively simple , by 4.5.7 .
From this in particular it follows that strongly effectively

simple and effectively simple are not equal notions. A further Theorem on strongly effectively simple sets will be shown later in point 8 .

c) A further property which implies T-completeness was found out from Lachlan .

A maximal set M is called effective maximal if there is a recursive function g , s.t. for all $e \geq 0$

$$(4.3) \qquad \mathrm{card}(\{ n : C_{W_e}(p_{\overline{M}}(n)) \neq C_{W_e}(p_{\overline{M}}(n+1)) \})$$

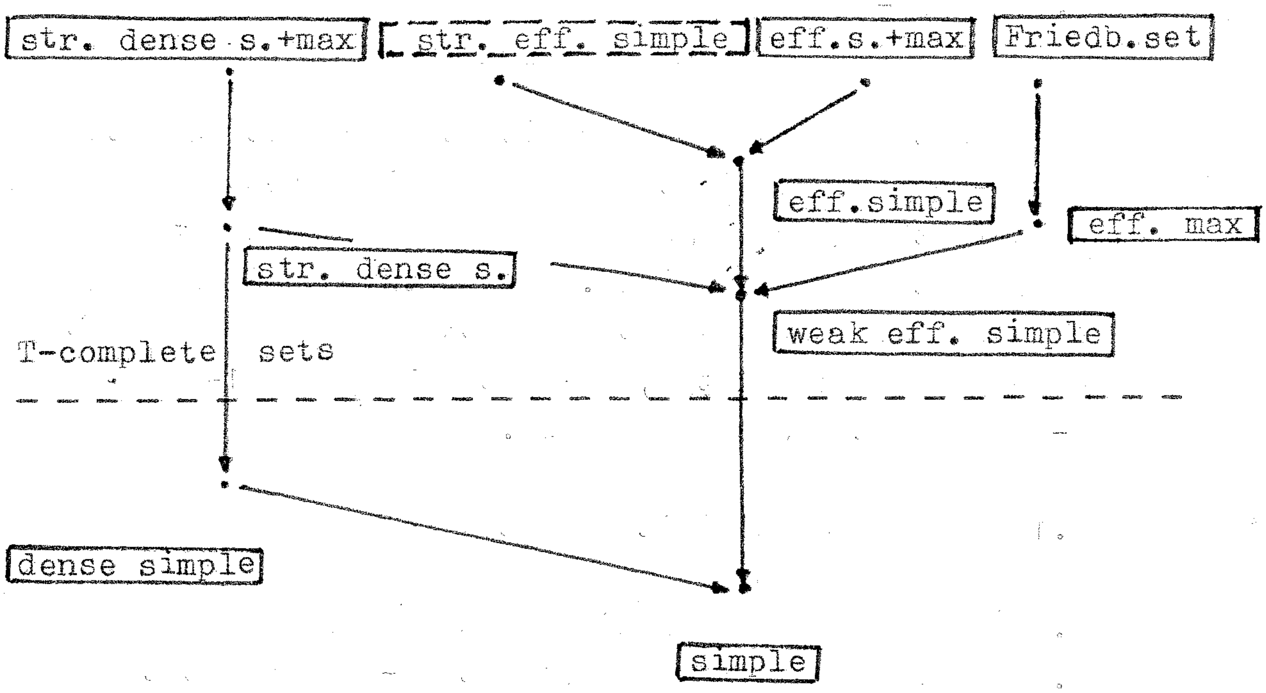
is bounded by g(e) .

Lemma 4.5.8 (Lachlan)/Od/,p.50: Every effective maximal set is T-complete .

Corollary 4.5.9: The maximal set constructed in Theorem 1.1.2 is T-complete .
Proof: The maximal set from 1.1.2 is effectively maximal , since (4.3) is bounded by a recursive function as in 1.1 it is announced .

Thus already the Friedberg construction gives a T-complete , maximal set . Lemma 4.5.8 will be remains true also if we claim only that g is recursive in M .

We have the following scheme about the notions considered here :



Thus we have five classes of T-complete , maximal sets , namely " strongly dense simple and maximal " , " effectively simple and maximal " , " Friedberg sets " , " effectively maximal " and the whole class " T-complete and maximal " (equal to " weak effectively simple and maximal ") . The mutually relationship between all these classes is not still complete analysed .

5. MANY - ONE REDUCIBILITY AND MAXIMAL SETS

Besides the Turing reducibility the connection between the many - one reducibility and the maximal sets was investigated. To this there are already known some results which concern different points of view . Inside this topic as a subpoint we analyse also the connections between the maximal sets and the one- one reducibility , in particulary the 1-degrees inside an m-degree with a maximal sets .

5.1 The m-reducibility and the m-degrees of maximal sets .

Definition 5.1.1: Let X and Y be subsets of ω . We say that X is many - one reducible to Y (symbolically : $X \leq_m Y$) if

$$(5.1) \qquad (\exists \text{ f rec. fct})(\forall x)(x \in X \longleftrightarrow f(x) \in Y) \quad .$$

The function f in (5.1) we call reducibility function (between X and Y) . We write $X \leq_m Y$ via f if $X \leq_m Y$ with the reducibility function f .

Basic properties of $\leq_m$ and others can be found e.g. in /Ro/ or /Od/ . Observe that an r.e. m-degree consists only of r.e. sets. Thus e.g. r.e.-minimal is equal with minimal and below O'_m .

The m-degrees with maximal sets have an easy to characterize position inside all r.e. m-degrees.

Theorem 5.1.2 (Young)/Ro,p.307/: Let A be a nonrecursive r.e. set and M a maximal set with $A \leq_m M$. Then $M \leq_m A$.

Corollary 5.1.3: The m-degrees of maximal sets are minimal.

Thus the positions of the m-degrees and of the T-degrees of maximal sets inside all r.e. m-degrees and all r.e. T-degrees respectively is extremely contrary.

With the same proof as for Theorem 5.1.2 a more general result can be proved.

Lemma 5.1.4: The m-degrees of $\mathcal{C}$- maximal sets are minimal.

Remarks to minimal r.e. m-degrees 5.1.5.

1. From 5.1.4 it follows that there are much more minimal r.e. m-degrees as those with maximal sets, since the class of $\mathcal{C}$- maximal sets includes many further r.e. sets, see 3.4.2,b). All these m-degrees except these with maximal sets includes only nonsimple r.e. sets. There are also minimal r.e. m-degrees without maximal sets but with simple sets, see /Od,p.62/.

2. What can be said about the class of m-degrees with maximal sets inside $\mathcal{Q}_{m,r.e.}$? In /Je,Lav/ it was shown that the class of all minimal r.e. m-degrees is bounded upwards only by $0'_m$. By the result mentioned in 4.3.3 this result can be improved. The class of all m-degrees with maximal sets are bounded upwards only by $0'_m$.
Equivalent to this is

$$(\forall X \text{ r.e.})((\forall M \text{ maximal})(M \leq_m X) \longrightarrow X \equiv K)) .$$

5.2. Reducibility functions between maximal sets . From Theorem 5.1.2 it follows immediately that two maximal sets M_1 and M_2 are m-equivalent iff there is a recursive permutation ρ with $\rho(M_1) =^* M_2$. It will be shown in this subpoint that all other reducibility functions between M_1 and M_2 are very similar to this ρ . Further we characterize the mutual position of the sequences $(p_{\overline{M}_1}(n))_{n \geqslant o}$ and $(p_{\overline{M}_2}(n))_{n \geqslant o}$ in the case that $M_1 \equiv_m M_2$.

Theorem 5.2.1 (Lerman)/Le,70/: Let M be a maximal set and f a recursive function with $f(\overline{M}) \cap \overline{M}$ infinite . Then $f|\overline{M} = Id|\overline{M}$ (mod Fin) (Id - identity mapping) .

Corollary 5.2.2: If M_1 and M_2 are maximal sets and f_1 , f_2 are reducibility functions between M_1 and M_2 , s.t. $f_1|\overline{R} = f_2|\overline{R}$.

Let X_1 and X_2 be two r.e. sets with $X_1 \leqslant_m X_2$.
We write $f_1 \approx f_2$ for reducibility functions f_1 , f_2 for X_1 and X_2 if

$$(\exists R_1 \text{ rec.})(\exists R_2 \text{ rec.})(R_1 \subset^* X_1 \wedge R_2 \cap X_1 =^* \emptyset \wedge \wedge f_1 = f_2 \text{ in } \overline{R_1 \cup R_2}) .$$

Obviously $f_1 \approx f_2$ is an equivalence relation . Corollary 5.2.2 says that for m-eqivalent maximal sets there exists exactly one equivalence class .

Lemma 5.2.3: If M_1 and M_2 are maximal sets and g is a partial recursive function with $g(\overline{M}_1) \subseteq \overline{M}_2$ and $g(\overline{M}_1)$ is infinite , then $M_1 \equiv_m M_2$ (also if $g(M_1) \not\subseteq^* M_2$) .

Let $(a_n)_{n \geqslant o}$ and $(b_n)_{n \geqslant o}$ be strongly increasing sequences . We say that $(a_n)_{n \geqslant o}$ and $(b_n)_{n \geqslant o}$ are almost alternating if there are numbers n_o and k_o , s.t.

$$(\forall l)(a_{n_o+1} < b_{k_o+1} < a_{n_o+l+1} < b_{k_o+l+1} < \dots) .$$

The following Theorem is an intensification of a result of Dëgtev /Dë,71/ .

Theorem 5.2.4: If A and B are maximal sets and $A \equiv_m B$ then $(p_{\bar{A}}(n))_{n\geqslant o}$ and $(p_{\bar{B}}(n))_{n\geqslant o}$ are almost alternating .

Let $A \equiv_{f.a.} B$ means that $(p_A(n))_{n\geqslant o}$ and $(p_B(n))_{n\geqslant o}$ are almost alternating . Then we have

$$A \equiv_m B \longrightarrow A \equiv_{f.a.} B \longrightarrow A \equiv_T B .$$

Both implications are properly already in the class Max .

Remark 5.2.5. If f is a recursive function with $f(n) \geqslant 1$ for all n and A and B r.e. sets with

$$\mathrm{card}(\{ x : x \in \bar{B} \wedge p_{\bar{A}}(n) \leqslant x < p_{\bar{A}}(n+1) \})$$

is equal to $f(n)$ for all n , then $A \equiv_T B$.

This is a good tool for constructing maximal sets A and B with $A \equiv_T B$ but $A \not\equiv_m B$. We get this if we take e.g. $f(n) = 2$ for all n , see later Theorem 5.5.1 .

5.3 Recursively enumerable sets m-equivalent to a maximal set.

Not very many is known about the r.e. sets which are m-equivalent to a maximal set . We will give here an introducting analyse of these classes of r.e. sets .

Let X be a r.e. set , M a maximal set and suppose $X \leqslant_m M$ via f. With S_n we denote the set $f^{-1}(\{n\})$, $n \geqslant o$. We shall see that the sequence $(S_n)_{n\geqslant o}$ is an important characterizing property of X .

We see at once that

X is q-maximal iff $(\exists k)(\forall x \in \bar{M})(\,|S_x| \leq k\,)$.

X is simple iff $(\forall x \in \bar{M})(\,|S_x| < \omega\,)$.

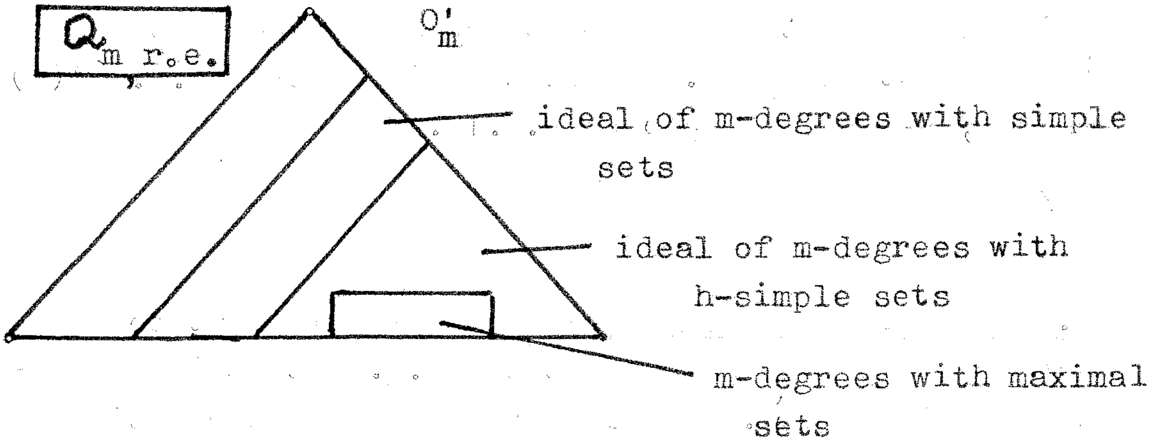
By using the notions introducted in 2.2 and 2.3 we can give a description of the simple sets m-equivalent to M .

<u>Theorem 5.3.1</u>: If A is h-simple , B simple and $B \leq_m A$ then B also is h-simple .

<u>Corollary 5.3.2</u> : 1) In an r.e. m-degree either all simple sets are not h-simple or all simple sets are h-simple .
2) The m-degrees with h-simple sets form an ideal in $\mathcal{Q}_{m,r.e.}$.
(Observe that X,Y h-simple $\longrightarrow$ $X \oplus Y$ h-simple) .

It is easy to see that every m-degree which is under a m-degree with a simple set also includes simple sets. Thus also the m-degrees with simple sets form an ideal in $\mathcal{Q}_{m,r.e.}$.

To it we can give the following picture :



The simple , not q-maximal sets which are m-equivalent to a maximal set are obviously not from $s\mathcal{HS}$, since the r.e. sequence $(\{x : (\exists! x_0,\dots,x_n)(\,x_0 < x_1 < x_2 < \dots < x_n < x \wedge f(x_i) = f(x)\,)\})_{n \geq 0}$

contradicts the definition of the elements of $s\mathfrak{H}\mathfrak{S}$.
The reltionship between these sets and the class $fs\mathfrak{H}\mathfrak{S}$ is more complicated.
From Lemma 2.1.2 it follows that in the case that the function $n \in \bar{M} \longrightarrow |S_n|$ is not bounded by a recursive function X is not from $fs\mathfrak{H}\mathfrak{S}$. That such sets X exist is easy to show. If $n \in \bar{M} \longrightarrow |S_n|$ is included in a recursive function (i.e. there is a recursive function g with $g(n) = |S_n|$ for $n \in \bar{M}$) than X is $fs\mathfrak{H}\mathfrak{S}$, by 2.1.2. This cannot be generalized to the case that $|S_n| \leq g(n)$ for $n \in \bar{M}$ and some recursive function g, since there is an $X \notin fs\mathfrak{H}\mathfrak{S}$, s.t. $|S_n| \leq n$ for all $n \geq 0$.

Problem: Are all simple, not q-maximal sets which are m-equivalent to a maximal set automorph ?

Generalization

A generalization of the class of r.e. sets m-reducible to a maximal set, but not q-maximal is the class

$$(5.2) \qquad \{X \in \mathcal{E}_{II} : (\forall i)(\ (X)_i \text{ is maximal in } (\omega \times \omega)_i\}.$$

In the following Theorem it is shown that the class (5.2) includes elements with new properties :

Theorem 5.3.4: There is a set A in the class (5.2), s.t.

$$(\forall X \in \mathcal{E}_{II}(A))(\forall i)(\ (X)_i =^* (\omega \times \omega)_i) \longrightarrow X =^* \omega \times \omega\ ).$$

Such a set of course cannot be m-equivalent with a maximal set.

5.4 1-degrees inside an m-degree with a maximal set. If we claim that the reducibility function in (5.1) is additionally injective we get a new reducibility. We regard here this new reducibility in particular inside the class of r.e. sets m-equivalent to a maximal set.

Definition 5.4.1: A set X is called <u>one-one reducible to Y</u> (X is 1-reducible to Y , symbolically : $X \leq_1 Y$) if there is a recursive and injective function f , s.t.

$$(\forall x)(x \in X \longleftrightarrow f(x) \in Y) \quad .$$

For our investigations it is better to work with a slight weakening of $\leq_1$.

We write $X \leq_1^* Y$ if

$$(\exists Z)(X \leq_1 Z \wedge Z =^* Y)$$

and $X \equiv_1^* Y$ for $X \leq_1^* Y \wedge Y \leq_1^* Z$.
The connection between $\leq_1$ and $\leq_1^*$ can be easily find out .

Let M be a maximal set , X an r.e. set with $X \leq_m M$ via f and Y an r.e. set $Y \leq_m M$ via g . When holds $X \leq_1^* Y$?

Let $S_n = f^{-1}(\{n\})$, $n \geq 0$ and $T_n = g^{-1}(\{n\})$, $n \geq 0$. By using Theorem 5.2.1 we can show

Theorem 5.4.2: If X and Y are simple , then $X \leq_1^* Y$ iff

$$(\forall \text{ a.a. } x \in \bar{M})(|S_x| \leq |T_x|) \quad .$$

Remark 5.4.3. If X and Y are arbitrary r.e. sets m-equivalent to M then

$$X \leq_1^* Y \longleftrightarrow (\forall \text{a.a. } x \in \bar{M})(|S_x| \leq |T_x| \wedge ((\exists x \in \bar{M}) (|S_x| \text{ is infinite } \longleftrightarrow (\exists x \in \bar{M}) (|T_x| \text{ is infinite })) . \tag{5.3}$$

Interesting in 5.4.2 and 5.4.3 is that the indizes in $(S_n)_{n \geq 0}$ and $(T_n)_{n \geq 0}$ are the same if $X \leq_1^* Y$ (otherwise 5.2.1 would be condradicted) .

The sublattice of $\mathcal{E}_{II}$ of initial sets

By using of 5.4.3 we can characterize by means of lattice theoretic notions the $\leqslant_1$ structure of an m-degree with a maximal set.
For this we define a special sublattice of $\mathcal{E}_{II}$.

Definition 5.4.4: A set X from $\mathcal{E}_{II}$ is called initial set if

$$(\forall i)(\forall x)(\ x \in (X)_i \longrightarrow (\forall y)(\ y \leqslant x \longrightarrow y \in (X)_i\ )\ ) .$$

Since the initial sets are closed under $\cap$ and $\cup$ this subclass forms a sublattice of $\mathcal{E}_{II}$. We denote this lattice with $\mathcal{E}_{in}$

If X is a r.e. set $(X \in \mathcal{E})$, then by $\mathcal{E}_{in}(X)$ we denote the lattice

$$\{ Y \in \mathcal{E}_{in} :\ (\forall i)(i \in X \longrightarrow (\forall x)(\ (x,i) \in Y\ )\ ) \} .$$

Let M be a maximal set. For $X \in \mathcal{E}_{in}(M)$ and $Y \in \mathcal{E}_{in}(M)$ let $X \approx_i Y$ holds if

> There are columns $Z_1, \dots, Z_n$ (see 2.3.2), s.t.
> for $Z = \overline{Z_1 \cup \dots \cup Z_n}$ holds $X \cap Z = Y \cap Z$ and
> $(\exists\ Z \text{ column})(\ Z \nsubseteq M \wedge Z \subseteq X\ ) \longleftrightarrow$
> $(\exists\ Z \text{ column})(\ Z \nsubseteq M \wedge Z \subseteq Y\ )$.

Theorem 5.4.5: $(\mathcal{E}_{in}(M), \subseteq)/_{\approx_i} \cong ([M]/_{\equiv_1^*}, \leq_1^*)$.

Characterization of $([M]_m/_{\equiv_1^*}, \leq_1^*)$

The structure of this poset , as it is to conjecture , is very difficult.

1° Interesting is that this poset is even a distributive lattice. This follows from the fact that $\approx_i$ is lattice definable (even elementary definable) in $\mathcal{E}_{in}(M)$.

2° The class of elements of $[M]_{m/\equiv_1^*}$ having a simple set correspondences to the subclass of $\mathcal{E}_{in}(M)$ consisting of all sets X from $\mathcal{E}_{II}$ with

$$(\forall i)(\ (\forall x)(\ (x,i) \in X \longrightarrow i \in M\ )(\text{mod} =^*)\ ) \quad .$$

Thus $(X)_i$ is finite for all $i \notin M$. Denote this subclass which forms an ideal with I . We get

$$(\mathcal{E}_{in}(M), \subseteq)_{/I} \cong \Pi^2 \quad (\text{ the lattice of all } \Pi^0_2 \text{ sets under inclusion }) \quad .$$

3° It is easy to see that for every $X \in I$

$$(I, \leq) \cong (\{Y \in I : X \leq Y\}, \leq) \quad .$$

4° An subclass of I is formed by the following elements :

$$(\exists g \text{ rec. fct})(\forall i)(\forall x)(\ i \notin M \longrightarrow (X)_i = [0, g(i)]\ )$$

(the sets determined by g , for $i \notin M$) .

By the maximality of M we get that this sets (mod $=^*$) are linear ordered by $\leq$ with the order type

$$\omega + \square(\eta, \omega^+ + \omega) \quad ,$$

where η is the order type of the rational numbers , $\omega^+ + \omega$ the order type of all integers and $\square(\cdot,\cdot)$ the shuffling operator .

But this is only a weak characterization , since there are still elements of I laying between the elements bounded by a recursive function.

1-degrees of $\mathcal{C}$ - maximal sets

Since maximal sets are simple , the 1-degrees of maximal sets are not minimal . But

Lemma 5.4.6: If A is $\mathcal{C}$ - maximal , not maximal then A has a minimal 1-degree .

Thus in particular if (A,B) is a Friedberg splitting of $A \cup B$ and $A \cup B$ is maximal , A and B have minimal 1-degree. Further using Theorem 5.2.1 it can easily shown that both 1-degrees are incomparable , see also /Od,p.67/ and even incomparable m-degrees .

5.5 m-degrees with maximal sets inside a high degree . Let d be a high degree. By Theorem 4.2.4 we know that d includes maximal sets. In the following we investigate the number of not pairwise m-equivalent maximal sets with the T-degree d . Yates showed in /Ya,69/ that for $d = 0_T'$ there are infinitely many m-degrees with maximal sets.which all belongs to d. A general answer to this problem was given by Lerman in /Le,70/ .

Theorem 5.5.1 (Lerman): Every high degree includes infinitely many m-degrees with maximal sets.

By using the Remark 5.2.5 the original proof in /Le,70/ can be simplified .

Remark 5.5.2. Every high degree includes still much more r.e. m-degrees as those with maximal sets. Jockusch showed e.g. in /Jo,69/ that every r.e. T-degree , not recursive , includes an infinite chain of r.e. m-degrees (i.e. $m_1 <_m m_2 <_m \dots$ all in the same T-degree) . At most m_1 can ~~has~~ have maximal sets, by Theorem 5.1.2 .
The structure of r.e. m-degrees inside an r.e. T-degree probably will be not easier than the whole structure $\mathcal{Q}_{m,r.e.}$. See for this /Od/ .

Classification of maximal sets

Regarding the three binary relations $\cong_{eff}$, $\equiv_T$ and $\equiv_m$ together in side the class Max it can be conjectured that for two maximal sets M_1 , M_2 the intersection of

$$\{ M \text{ max.} : M \equiv_T M_1 \} \quad , \quad \{ M \text{ max.} : M \cong_{eff} M_2 \}$$

is not empty , more , includes infinite many maximal sets pairwise not $\equiv^*$- equivalent .

6. FURTHER REDUCIBILITIES AND MAXIMAL SETS

Besides the both reducibilities and their relationship to the maximal sets investigated in the two preceding points it was also still analysed the connections between the maximal sets and other reducibilities. Above all the tt-reducibility is to mention here . But we shall see that the position of the degrees with maximal sets inside all r.e. degrees is not so clearly determined as before and the actual knownlegde about it is not complete. Nevertheless few characterizating properties are known. These results will be stated in this point .

6.1 tt-reducibility , definition and equivalent formulations
6.2 tt-reducibility and maximal sets
6.3 Weak and strong modifications of the tt-reducibility

6.1 tt-reducibility , definition and equivalent formulations .

Definition of the tt-reducibility

A tt-reducibility is a pair $<\alpha, <a_1,\ldots,a_n>>$ with α an n-ary Boolean function (i.e. $\alpha : I^n \longrightarrow I$, $I = \{0,1\}$) and $<a_1,\ldots,a_n>$ an n-tuple of numbers .

We can assign effectively and bijective to every tt-requirement which is assigned to the number x .

(We say that $tt_x = <\alpha, <a_1,\ldots,a_n>>$ is satisfied in the set X , if $\alpha(C_X(a_1),\ldots,C_X(a_n)) = 1$ and write in this case $X \models tt_X$.

Definition 6.1.1: Let X and Y be two sets . We say that X is truth - table reducible to Y (tt-reducible , symbolically: $X \leqslant_{tt} Y$) if

$$(\exists F \text{ rec. fct})(\forall x)(x \in X \longleftrightarrow Y \models tt_{F(x)}) \quad .$$

Remarks. The relation $\leq_{tt}$ is reflexive and transitive /Ro,p.146/ and thus is an reducibility .
This reducibility lays properly between $\leq_m$ and $\leq_T$. Basic properties of $\leq_{tt}$ can be found e.g. in /Ro/ or /Od/ .

If $tt_x = \langle \alpha, \langle a_1,\dots,a_n \rangle \rangle$, then $\{tt_x\}$ means the set $\{a_1,\dots,a_n\}$.

Oftenly instead of Boolean functions the tt-requirements are formulated by means of sentential formulas. Since every sentential formula φ corresponds effectively in a unique way to a Boolean function $\alpha\varphi$ and for every Boolean function α we can find effectively a formula φ with $\alpha = \alpha\varphi$. Both formulations are equivalent.

Equivalent formulations of the tt-reducibility

We give here still other equivalent formulations to the tt-reducibility . By means of these the position of the tt-reducibility to other reducibilities will be become clearer. Further on instead of the definition oftenly in the Theorems one of the equivalent formulations is used .

6.1.2. It holds the following equivalence given by Shoenfield /Sh,p.193/ :

$$X \leq_{tt} Y \longleftrightarrow (\exists F \text{ rec. fct})(\exists P \text{ rec. relation}) \\ (\forall x)(x \in X \longleftrightarrow P(\sigma(Y,F(x)))) \quad .$$

6.1.3. The following formulation of Marczenkov /Mar/ is a small variation of 6.1.2 :

$$X \leq_{tt} Y \longleftrightarrow (\exists F \text{ rec. fct})(\exists G \text{ rec. fct}) \\ (\forall x)(C_X(x) = G(\sigma(Y,F(x)))) \quad .$$

Take in 6.1.2 instead of P the characteristic function of P and place it for G in 6.1.3 . The formulation 6.1.3 shows good the connection with some other reducibility ,see 6.3 .

6.1.4 This equivalence was given by Nerode /Ro,p.187/ :

$$X \leqslant_{tt} Y \longleftrightarrow (\exists e)(\forall Z)(\ \varphi_e^Z \text{ is total } \wedge C_X = \varphi_e^Y\ )\ .$$

Here we see good that $\leqslant_{tt}$ is included into $\leqslant_T$, since for $\leqslant_{tt}$ we require additionally that φ_e^Z is total for all sets Z.

6.1.5 An other equivalent formulation is given in /Ro,p.203/.

$$\begin{aligned} X \leqslant_{tt} Y \longleftrightarrow\ & (\exists F \text{ rec. fct})(\exists G \text{ rec. fct})(\forall x) \\ & (\ x \in X \longleftrightarrow (\exists u)(\exists v)(\ \langle u,v\rangle \in D_{F(x)} \wedge \\ & \qquad\qquad D_u \subseteq Y \wedge D_v \subseteq \bar{Y}\ )\ ) \wedge \\ & (\ x \notin X \longleftrightarrow (\exists u)(\exists v)(\ \langle u,v\rangle \in D_{G(x)} \wedge \\ & \qquad\qquad D_u \subseteq Y \wedge D_v \subseteq \bar{Y}\ )\ )\ )\ . \end{aligned}$$

If we replace in (6.1) " $\langle u,v\rangle \in D_{F(x)}$ " by " $\langle u,v\rangle \in W_F(x)$ " and " $\langle u,v\rangle \in D_{G(x)}$ " by " $\langle u,v\rangle \in W_{G(x)}$ " , then we get an equivalent requirement for $X \leqslant_T Y$.

<u>6.2 tt-reducibility and maximal sets</u> . If M_1 and M_2 are maximal sets with $M_1 \equiv_{tt} M_2$ then there exists a reduction algorithm between both which has a very easy structure as in /Ko,73b/ it was shown .

<u>Theorem 6.2.1 (Kobcev):</u> If M_1 and M_2 are maximal sets then $M_1 \equiv_m M_2$ iff $M_1 \equiv_{tt} M_2$.

Thus in the case $M_1 \equiv_{tt} M_2$ there exists already a recursive permutation ρ with $\rho(M_1) =^* M_2$, see 5.2.2 .
Theorem 6.2.1 means that the degree $[M]_{tt}$ with M maximal includes exactly one m-degree with maximal sets . Thus every

degree from $\mathbb{H}^1$ includes infinitely many tt-degrees with maximal sets , by 5.5.1 . Applying 5.2.4 we see that a high degree includes even an infinite antichain of tt-degrees with maximal sets .

Problem : Holds for maximal sets M_1 and M_2 even the equivalence: $M_1 \leq_{tt} M_2$ iff $M_1 \leq_m M_2$?

tt-degrees of maximal sets

We give here the already known facts about the tt-degrees with maximal sets inside all r.e. tt-degrees .

Completeness. At first we shall see that maximal sets are not tt-complete .

A r.e. set X is called recursively separable (for short : r-separable) if

$$(\forall Y \text{ r.e.})(X \cap Y = \emptyset \longrightarrow (\exists R \text{ rec.})(X \subseteq R \wedge R \cap Y = \emptyset)).$$

We say that X is r-inseprable if X is not r-separable .

Theorem 6.2.2 (Denisov)/De,74/: If A is an r-separable r.e. set and $A \leq_{tt} B$ then B is not h-simple .

Corollary 6.2.3 (Post)/Po/: A tt-complete set is not h-simple

Proof: If (B_1,B_2) is a Friedberg splitting of the nonrecursive r.e. set $B_1 \vee B_2$, then B_1 (and so also B_2) is r-inseparable , see /Ro,p.318/. Thus there are r-inseparable sets. But this means that tt-complete sets cannot be h-simple by Theorem 6.2.2 .

Thus h-simple sets are not tt-complete . But there are simple sets which are tt-complete , see /Ro,p.148/ .

Corollary 6.2.4: Maximal sets are not tt-complete .

Minimality. The next natural question on the position of the tt-degrees with maximal sets is if they can be very small. We shall see that this also is inpossible. Observe that not every tt-degree below $0'_{tt}$ includes r.e. sets . Thus it is necessary to distingwish between " minimal and below $0'_{tt}$ " and " minimal among all r.e. tt-degrees " . Are these both notions equivalent or not ?

The question if tt-degrees with maximal sets can be minimal among all r.e. tt-degrees can be answered using Corollary 4.2.2 and the following Theorem :

Theorem 6.2.5 (Dëgtev)/Dë,78/: If the r.e. set X has r.e.-minimal tt-degree then X is not high.

Corollary 6.2.6: A maximal set has no r.e.-minimal tt-degree.

Thus the tt-degrees of the maximal sets lay in the " middle " inside all r.e. tt-degrees.

Incomparability. Regarding the set of all tt-degrees with maximal sets as already in 4.3.3 mentioned a property follows from the Theorem

Theorem 6.2.7 (Marchenkov): If d is a r.e. tt-degree not 0 and not $0'_{tt}$ then there is a maximal set M , s.t. d and $[M]_{tt}$ are incomparable .

Corollary 6.2.8: The tt-degrees of maximal sets are bounded upwards only by $0'_{tt}$ (inside the r.e. tt-degrees)

From Theorem 4.3.1 follows that the tt-degrees even of two special maximal sets are bounded down wards only by 0 .

Can we find a maximal set as in Theorem 6.2.7 having a given high T-degree ?

6.3 Weak and strong modifications of the tt-reducibility .

By relatively small and natural modifications of the tt-reducibility we get new reducibilities which are different already in the class of complete sets. Thus it is sensfull to investigate the relationship between these new reducibilities and the maximal sets .

Definition 6.3.1: We say that X is bounded truth-table reducible to Y ($X \leq_{btt} Y$) if $X \leq_{tt} Y$ via f and $|\{tt_{f(x)}\}| \leq n$ for all x .

The smallest such n is called norm of the reducibility between X and Y .

It is easy to see that in the case $X \leq_{btt} Y$ via f we can claim that $|\{tt_{f(x)}\}| = n$ for all x . If $\{tt_{f(x)}\} = \{a_1, \dots a_k\}$ with $k < n$, take new numbers $a_{k+1}, \dots, a_n$ and define $\alpha_0(i_1, \dots, i_k, i_{k+1}, \dots, i_n) = \alpha(i_1, \dots, i_k)$ for all $i_{k+1}, \dots, i_n$.

Definition 6.3.2: We say that X is weak truth-table reducible to Y . ($X \leq_{wtt} Y$) , if $X \leq_T Y$ and there is a recursive function f , s.t. for every x to decide if $x \in X$ or not we need only the informations $n \in Y$ or not for $n \leq f(x)$, i.e.

$$X \leq_{wtt} Y \longleftrightarrow (\exists f \text{ rec. fct})(\exists e)(\ C_X(n) = m \longleftrightarrow (\exists y \leq f(n))(\ R(\sigma(Y,y),e,n,m)\ )\ ,$$

where R is the recursive relation from 4.1 .

An other equivalent formulation to $X \leq_{wtt} Y$ is that for deciding " $x \in X$ " we need only the informations " $n \in D_{f(x)}$. Since

$$(\exists f \text{ rec. fct})(\forall x)(D_x \subseteq \{0,1,\ldots,f(x)\})$$

and

$$(\exists h \text{ rec. fct})(\forall x)(\{0,1,\ldots,x\} = D_{h(x)})$$

This defines the same reducibility.

Remark. In the definition of $X \leq_T Y$ there also exists a function f , s.t. only $y \leq f(x)$ are used for calculating if $x \in X$ or not . But here f in general is not recursive, but only recursive in Y .

The following formulation of $\leq_{wtt}$ in /Mar/ shows the connection and also the differences with $\leq_{tt}$:

$$X \leq_{wtt} Y \longleftrightarrow (\exists \varphi \text{ rec. fct})(\exists \psi \text{ par.rec. fct}) (\forall x)(C_X(x) = \psi(\sigma(Y, \varphi(x)))) .$$

Between all in this paper regarded reducibilities there hold the following inclusions :

$$\boxed{\leq_1 \longrightarrow \leq_1^* \longrightarrow \leq_m \longrightarrow \leq_{btt} \longrightarrow \leq_{tt} \longrightarrow \leq_{wtt} \longrightarrow \leq_T} .$$

At first we observe that Theorem 6.2.1 can be generalized to $\leq_{wtt}$.

Corollary 6.3.3: Let M_1 and M_2 be maximal sets . Then $M_1 \equiv_m M_2$ iff $M_1 \equiv_{wtt} M_2$.

Hence every high degree d includes infinitely many wtt-degrees with maximal sets. For all other reducibilities (except $\leq_1^*$)

a degree with a maximal set includes only one degree with a maximal set of a preceding reducibility .

Theorem 6.3.4: Let M_1 and M_2 be maximal sets. Then $M_1 \leq_m M_2$ iff $M_1 \leq_{btt} M_2$.

Thus for $\leq_{btt}$ we have an answer to the problem of 6.2.

Complete sets . By using results on btt-complete and wtt-complete sets it can easily be concluded from it that maximal sets have not these properties .

That maximal sets are not btt-complete follows already from the result about tt-completeness 6.2.3 . For btt-complete sets it holds even a stronger property

Theorem 6.3.5 (Post)/Ro,p.151/: A btt-complete set is not simple .

Theorem 6.2.2 can be generalized to $\leq_{wtt}$. Thus we have

Corollary 6.3.6: The wtt-complete sets are not h-simple .

Corollary 6.3.7: The maximal sets are not wtt-complete

The m-complete sets and btt-complete sets do not coincide , since every wtt-complete set includes even infinitely many r.e. m-degrees , see /Od/ . The tt-complete sets and wtt-complete sets do not coincide , since a wtt-complete set includes at least two r.e. tt-degrees /Od,p.82/ . Thus all considered reducibilities from $\leq_m$ to $\leq_T$ are different already inside the classes of their complete sets.

Minimality. Since not every btt-degree below O'_{btt} includes r.e. sets , see /Od,p.79/ we have also here to distinguish between " minimal and below O'_{btt} " and " r.e. - minimal " .

Theorem 6.3.8 (Kobcev)/Ko,73a/: Maximal sets have no r.e.-minimal btt-degree.

Corollary 6.3.9: Let M be a maximal set. Then $[M]_{tt}$ includes at least two r.e. btt-degrees (and so at least two r.e. m-degrees) .

Proof: Since Marchenkov has proved /Od,p.79/ that every r.e. tt-degree includes a minimal r.e. btt-degree , by Theorem 6.3.8 there must be still an other r.e. btt-degree .

Remark. The analogous Theorem to 6.2.7 holds also for $\leqslant_{btt}$ and $\leqslant_{wtt}$, as in 4.3 already mentioned .

Except the reducibilities regarded here there are many other which are different already inside the class of r.e. sets . Such an reducibility is e.g. $\leqslant_Q$. The connections between this and Max also could be analysed . Already known is that maximal sets are not Q-maximal , see /Od,p.49/ . An other connection between the maximal sets and $\leqslant_Q$ is announced in /Om/ .

7. POSITIVE - GENERITY AND MAXIMAL SETS

From the set theory , more precisely from the Forcing theory it was taken over to the recursion theory the notion of generic set and applied to recursion theoretic investigations. Here we shall work with a modification of generic sets , done by Jockusch , and use it to the analyse of r.e. sets . In this point we present above all , besides the basic definition , the results concerning the maximal sets. The main paper for this is the paper of Ingrassia /In/ . General for all r.e. sets there was also investigated by Maass , see /Ma,82/.
For the definitions and representations of the new results we use the finite 01-sequences in the meaning of iv) mentioned in the Introduction . We shall see that just the Σ^0_1 ideals of finite 01-sequences and thus the ideal lattice $\mathcal{E}_I$ have a fundamental meaning for this topic .

7.1 Σ^0_1 - relations with a set variable
7.2 Positive-generic recursively enumerable sets
7.3 (m,n) - generic - generic recursively enumerable sets

7.1 Σ^0_1 - relations with a set variable . In this subpoint we will analyse the recursively enumerable relations with set variables and show the correspond of these with Σ^0_1 ideals in $2^{<\omega}$.

A relation P(X) is rec. enum. if there is a 2-ary recursive relation R , s.t. for every $X \in \mathcal{P}(\omega)$

$$P(X) \longleftrightarrow (\exists n)(R(\widehat{\sigma}(X,n),n) ,$$

see e.g. /Se,67/ , p. 162 .

We denote this class of relations with $\Sigma^0_{1,0}$.

7.1.1 Arithmetical representation of relations from $\Sigma^0_{1,0}$

Theorem 7.1.1.1: All formulas of the form

$$(7.1) \qquad (\exists x_1)(\exists x_2)\ldots(\exists x_n)\,\varphi\,(x_1,\ldots,x_n,X)$$

where φ includes only bounded quantifies , give as classes of sets satisfying (7.1) exactly all $\Sigma^0_{1,0}$-relations .

7.1.2 Representation of $\Sigma^0_{1,0}$ - relations as ideals of $2^{<\omega}$

All $\Sigma^0_{1,0}$-relations can be also represented in an equivalent way as ideals of $\mathcal{E}_I$. It holds

Theorem 7.1.2.1: Every relation P from $\Sigma^0_{1,0}$ corresponds in a unique way with an ideal Δ from $\mathcal{E}_I$ by

$$(\forall X)(\ P(X) \longleftrightarrow (\exists n)(\sigma(X,n) \in \Delta\ )$$

and for every $\Delta \in \mathcal{E}_I$ there is such relation P .

7.1.3 Representation of special lattices as ideal lattices

Denote with $\mathcal{E}^1_{1,0}$ the lattice of all $\Sigma^0_{1,0}$ - relations together with $\wedge$ and $\vee$.

Corollary 7.1.3.1: The lattices $\mathcal{E}^1_{1,0}$ and $\mathcal{E}_I$ are isomorphic

Denote with $\mathcal{E}^1_{1,1}$ the lattice of all $\Sigma^0_{1,1}$- relations with one set variable and one individual variable . Thus $P \in \mathcal{E}^1_{1,1}$ has the definition

$$(\forall X)(\forall x)(\ P(X,x) \longleftrightarrow (\exists n)R(\sigma(X,n),n,x)\ )$$

Let $\Delta_0 = \{\mathfrak{z} \in 2^{<\omega} : \mathfrak{z} \neq 0\ldots0 \vee \mathfrak{z} \neq \emptyset\}$ and $\mathcal{E}_I|\Delta_0$ the sublattice $\{\Delta \in \mathcal{E}_I : \Delta \leq \Delta_0\}$.

Lemma 7.1.3.2: $\mathcal{E}^1_{1,1}$ and $\mathcal{E}_I|\Delta_0$ are isomorphic .

Remark. Observe that $\mathcal{E}^1_{1,1}$ and $\mathcal{E}^1_{1,0}$ are not isomorphic , since $\mathcal{E}_I$ and $\mathcal{E}_I|\Delta_0$ are not isomorphic .

7.2 Positive - generic recursively enumerable sets . The notion of positive genric set is a modification of the so-called 1-generic set . First we treat the 1-generic sets and get from this notion the notion of positive generic set .

Definition 7.2.1 : A subset Z of ω is called 1-generic if for every P from $\Sigma^0_{1,0}$ holds

$$(7.2)\quad \neg P(Z) \longrightarrow (\exists \mathfrak{z} \in 2^{<\omega})(\forall X)(\mathfrak{z} \in \mathfrak{S}(X) \longrightarrow \neg P(X)) .$$

If P(Z) holds , then of course also

$$(\exists \mathfrak{z} \in 2^{<\omega})(\forall X)(\mathfrak{z} \in \mathfrak{S}(X) \longrightarrow P(X)) ,$$

what easily follows from Theorem 7.1.2.1 .

If Z is 1-generic and $\Delta \in \mathcal{E}_I$, then it holds

$$(7.3)\quad Z \in \Delta \text{ or } (\exists \mathfrak{z})(\mathfrak{z} \in \mathfrak{S}(Z) \wedge \mathfrak{T}[\mathfrak{z}] \cap \Delta = \emptyset) .$$

From this follows that 1-generic sets are infinite and also coinfinite .

Theorem 7.2.2 : There are 1-generic sets

Lemma 7.2.3: Recursively enumerable sets are not 1-generic

It is easy to construct an ideal $\Delta \in \mathcal{E}_I$, s.t. (7.3) is not satisfied.

Thus the notion of 1-generic set is not useful for the theory of r.e. sets. This leads to the following modification :

Definition 7.2.4 (Jockusch): A subset Z of ω is called positive - generic (shortly , p-generic) if Z is coinfinite and for every $P \in \Sigma^0_{1,0}$ holds

$$(7.4) \quad \neg P(Z) \longrightarrow (\exists H \text{ r.e. subset of } Z)(\exists Y \text{ finite subset of } \bar{Z})(\forall X)(H \subseteq X \wedge Y \subseteq \bar{X} . \longrightarrow \neg P(X)).$$

If Z is an r.e. set , then we can put in (7.4) H equal to Z .

There are p-generic r.e. sets. But for us the comparison with the class Max is of interest . It holds

Theorem 7.2.5 (Ingrassia)/In/: The both notions " maximal set " and " p-generic r.e. set " are independent (i.e. all four combinations of these notions and their negations are not empty).

7.3 (m,n) - positive - generic sets . In /In/ the notion p-generic was weaken . These weakens we can get when in the definition 7.2.4 not all Σ^0_1 - relations are taken , but only special subclasses . We shall see that these weaker notions are meaningfull for the analyse of the maximal sets .

A list is an r.e. set of pairs (D_n, D_m) with $D_n \cap D_m = \emptyset$. (The definition of D_n is given in 2.2.4). Thus every list L can be presented in the form

$$\{(D_n, D_m) : \langle n,m \rangle \in W_e\}$$

for some index $e \geq 0$.

We say that a list L is a <u>(k,l)-list</u> if for all pairs (D_n, D_m) from L hold $|D_n| \leq k$ and $|D_m| \leq l$.

With every list L there is connected an Σ^0_1-ideal Δ_L in the following way : Let (D_n, D_m) belongs to L and $\tau \in 2^{<\omega}$ be, s.t.

$$|\tau| = \max (D_n \cup D_m) \quad , \quad \begin{array}{l} \tau(i) = 1 \longleftrightarrow i \in D_n \\ \tau(j) = 0 \longleftrightarrow j \in D_m \end{array} .$$

Δ_L is the smallest ideal including $\mathfrak{I}[\tau]$ with τ defined as above (resp. to an arbitrary pair of L).

<u>Definition 7.3.1</u>: We say that a set X is <u>(k,l)-p-generic</u> if X is coinfinite and p-generic resp. to all (k,l)-lists (i.e. (7.4) is satisfied for all $P \in \Sigma^0_{1,0}$ corresponding to Δ_L (by Th. 7.1.2.1) and L is an (k,l)-list).

<u>Definition 7.3.2</u>: A set X is called <u>$(<\infty,l)$-p-generic</u> if X is (k,l)-p-generic for all $k < \omega$. (Analogously $(k, <\infty)$-p-generic and $(<\infty, <\infty)$-p-generic).

Since for every Σ^0_1-ideal there is a list L, s.t. $\Delta = \Delta_L$, (∞, ∞)-p-generic coincides with p-generic.
Observe that there are lists with are not (k,l)-lists for any numbers k,l. Thus $(<\infty, <\infty)$-p-generic and (∞, ∞)-p-generic are not equal each to other by definition.
It is obvious that if X is (k,l)-p-generic and k',l' are numbers with $k' \leq k$ and $l' \leq l$, then X is (k',l')-p-generic. But it can be shown, see /In/ that (1,n)-p-generic is equivalent to $(<\infty, n)$-p-generic for all $n \geq 1$.

<u>Theorem 7.3.3 (Ingrassia)</u>: 1) All maximal sets are (1,1)-p-generic.
2) There are maximal sets which are not (1,2)-p-genric.

8. RECURSIVELY ENUMERABLE SETS AND MAXIMAL SUPERSETS

In this point we will analyse the relationship between the maximal sets and all other r.e. sets . At this we are interested in particular in the r.e. sets which have no maximal supersets . It will be given Theorems of existence of such r.e. sets and properties which imply that a r.e. set has no maximal superset. In 8.3 a Theorem is formulated which characterizes the T-degrees of r.e. sets without maximal supersets. This Theorem gives an interesting connection between T-degrees of r.e. sets and special lattice properties of these r.e. sets.
Further we analyse the degrees of maximal sets which are supersets of an r.e. set .

8.1 Constructions of r.e. sets without maximal supersets
8.2 Low and Semilow sets
8.3 Turing-degrees of r.e. sets and the structure of their r.e. supersets
8.4 Reducibility degrees of the maximal supersets of an r.e. set

8.1 Constructions of r.e. sets without maximal supersets .

After the construction of a maximal set by Friedberg it arised the converse question , namely if there are also simple sets which have no maximal supersets. In the beginning of the 60th. Martin gaves a construction of such simple sets. We beginn the subpoint with this result. Further it is included a result to this topic shown by Cohen and Jockusch .

An r.e. set without maximal supersets we call for short atomless . This does not lead to a misunderstanding , since the lattice $\mathcal{E}^*$ has no minimal elements .

Theorem 8.1.1 (Martin)/Ro,p.305/: There are atomless r.e. sets.

The method of the proof of Theorem 8.1.1 can be used also for showing many other properties of r.e. sets , e.g.

- the class of full r.e. sets is not cofinal in $\mathcal{E}$
- All elements of $[M]_m$, see 5.3 , which are not q-maximal have atomless r.e. supersets .

Corollary 8.1.2 (Martin)/Ro,p.306/: Every coinfinite , not h-simple r.e. set has h-simple , atomless supersets as well as non h-simple , atomless supersets.

Remark. The construction method of Theorem 8.1.1 can also be applied to a greater class as the maximal sets, namely e.g. to the hh-simple sets . We can do this if we use the following criterion for hh-simplicity of Martin /Ma,63/:

> " A coinfinite r.e. set X is not hh-simple , iff there is a recursive array of disjoint sets $(S_n)_{n \geqslant 0}$, s.t.
>
> $$\text{card}(\{ n : |S_n \cap \overline{X}| \geqslant n \}) \text{ is infinite}$$ " .

The relationship between dense simple sets and strongly effectively simple sets was already investigated in 3.5 . Here we give a further result to this topic from which we can conclude Corollaries interesting also for the analyse of the maximal sets.

Theorem 8.1.3 (Cohen,Jockusch)/Co,Jo/: Every coinfinite and not dense simple r.e. set is contained in a strongly effective simple set.

Corollary 8.1.4 (Cohen,Jockusch): The Post's simple set is atomless .

Proof: The Post's simple set is strongly effective simple. Thus every coinfinite r.e. superset is it.

By Theorem 4.5.6 no r.e. superset can be maximal .

Corollary 8.1.5: Every coinfinite r.e. set X has a r.e. superset of high degree and if X is not high, then even a T-complete r.e. superset.

Corollary 8.1.6: If X is a coinfinite r.e. set and $\mathcal{DS}$ is cofinal in $\mathcal{E}(X)$, then X already is dense simple.

Remark. Although maximal sets are not strongly effectively simple there are already r-maximal sets which are .

8.2 Low and Semilow sets . For the topic of investigation of this point , not as in point 4 the high degrees are of interest but the degrees low_1 and low_2 as we shall see. Since many results shown for low_1 or low_2 sets can be generalized to a greater class of r.e. sets , we introduce here these generalizations which are called semilow sets. In this subpoint the definition of these greater classes and their relationship to the low sets will be given .

Semilow sets

Let X be a subset of ω . With NE_X (NE - Not Empty) we denote the set $\{ e : W_e \cap X \neq \emptyset \}$ and with Fin_X the set $\{ e : W_e \cap X - \text{finite} \}$.

Definition 8.2.1 (Soare): A set X is called Semilow_1 if NE_X belongs to Δ^0_2 , $\text{Semilow}_{1.5}$ if Fin_X belongs to Σ^0_2 , Semilow_2 if Fin_X belongs to Δ^0_3 .

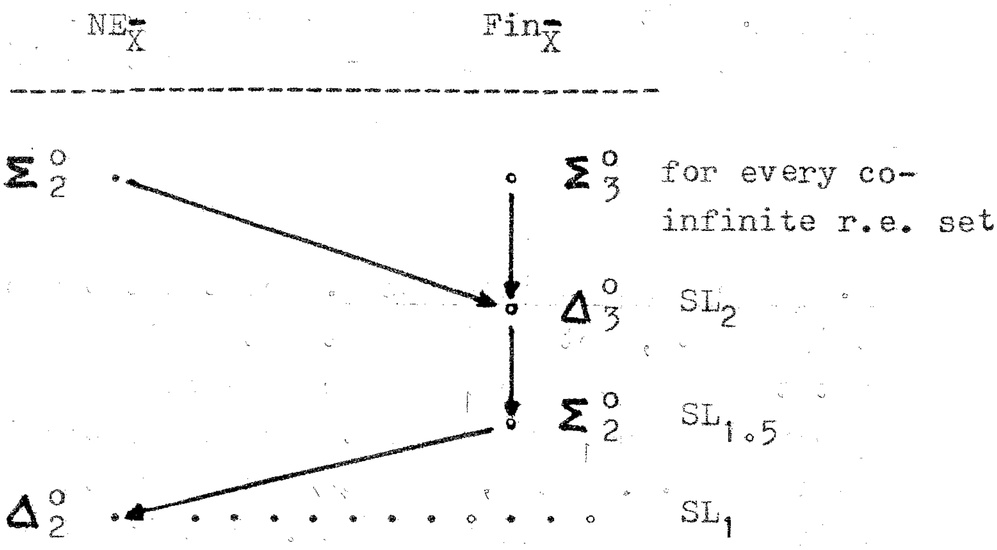
For the class of Semilow_x we use the symbol SL_x , x = 1,1.5,2.

Between these three classes the following (proper) implications hold :

$$X \in SL_1 \longrightarrow X \in SL_{1.5} \longrightarrow X \in SL_2 \quad .$$

Every r.e. set is obviously semilow$_1$. Thus only the classification of the complements of r.e. sets is significant.

Let X be a coinfinite r.e. set. Then $Fin_{\bar{X}}$ always belongs to Σ^0_3 and $NE_{\bar{X}}$ to Σ^0_2 . Thus we have the following scheme :



Observe that for every coinfinite r.e. set X $Fin_{\bar{X}}$ cannot belongs to Δ^0_2 , since every Σ^0_2 set is m-reducible to $Fin_{\bar{X}}$.

Remark. From Theorem 2.6.3 it follows that the class of non-speedable sets is just the class SL_1 .

The following Theorem shows that there are very many nonrecursive r.e. sets with the complement in SL_1 :

Theorem 8.2.2 (Soare)/So,74/: Every r.e. T-degree includes an r.e. set with the complement in SL_1 .

Comparision between low and semilow sets

The classes low_1 and low_2 , introduced in 4.1 are nearly connected with the classes SL_1 and SL_2 respectively . It holds

$$L_1 \longrightarrow SL_1 \ , \quad L_2 \longrightarrow SL_2 \ .$$

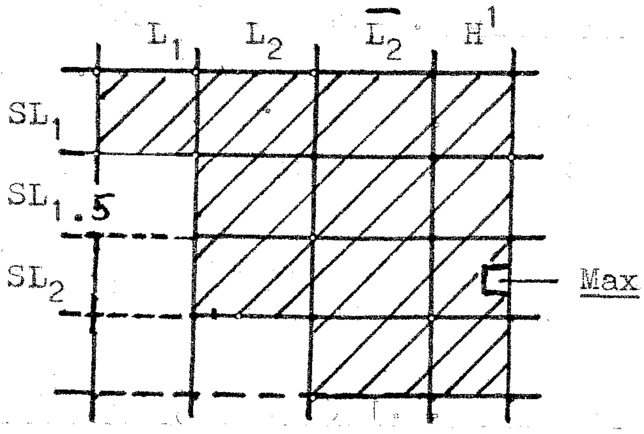
All here regarded classes of sets can be defined in uniform way . For this , besides the classes NE_X and Fin_X we need still the classes $NE = \{e : W_e \neq \emptyset\}$, $NE^X = \{e : W_e^X \neq \emptyset\}$, $Fin_I = \{e : W_e \text{ is finite}\}$, $Fin^X = \{e : W_e^X \text{ is finite}\}$.

$$\begin{array}{llll} L_1 : & NE^X \leq_T NE \ , & L_2 : & Fin^X \leq_T Fin_I \\ SL_1 : & NE_X \leq_T NE \ , & SL_{1.5} : & Fin^X \leq_1 Fin_I \\ & & SL_2 : & Fin_X \leq_T Fin_I \ . \end{array}$$

For more details of this characterization , see /Be,So/ .

Thus we have the following classification :

$$\begin{tikzcd} L_1 \arrow[r] \arrow[dd] & SL_1 \arrow[d] \\ & SL_{1.5} \arrow[d] \\ L_2 \arrow[r] & SL_2 \end{tikzcd}$$



Lattice properties of the $\text{semilow}_{1.5}$ sets

There was proved two Theorems on the $\text{semilow}_{1.5}$ sets which characterize the position of these sets inside $\mathcal{E}$.

Theorem 8.2.3 (Bennison,Soare)/Be,So/: The r.e. sets with semilow$_{1.5}$ complement are not finitely strongly h-simple sets .

Corollary 8.2.4: Maximal sets are from SL_2 , but not from $SL_{1.5}$ (More precisely , the complement is from SL_2 but not from $SL_{1.5}$)

The following Theorem gives a final answer to a problem investigated in several papers :

Theorem 8.2.5 (Maass)/Ma,83/: Let A be an infinite r.e. set . Then $\mathcal{E}(A)$ is effectively isomorphic to $\mathcal{E}$ iff $\overline{A}$ is semilow$_{1.5}$.

Corollary 8.2.6 (Bennison/Soare): Every coinfinite r.e. set with a semilow$_{1.5}$ complement has a maximal superset .

From 8.2.5 it follows obviously that a r.e. set from 8.2.6 has also always an atomless r.e. superset .

8.3 T-degrees of r.e. sets and the structure of their r.e. supersets . We show in this subpoint that there are connections between the T-degree of r.e. sets and the structure of their r.e. supersets. Above all the relationship between the r.e. supersets and the maximal sets will be investigated .

The following Theorem gives an interesting degree characterization of the atomless r.e. sets :

Theorem 8.3.1 (Lachlan,Shoenfield): A r.e. T-degree d includes an atomless r.e. set iff $d \notin L_2$.

In /La,68c/ it is shown that the degrees with atomless r.e.

sets belong to $\overline{L}_2$ and in /Se,76/ it is proved that every degree from $\overline{L}_2$ includes atomless r.e. sets .

The T-degrees of the r.e. sets having cofinal maximal supersets also can be characterized .

<u>Lemma 8.3.2</u>: A r.e. T-degree d includes a r.e. set with cofinal maximal supersets iff $d \in H^1$.
<u>Proof</u>: The implication $\longleftarrow$ is obvious by 4.2.4 . The other implication follows from 8.1.6.

Observe that the class of r.e. sets with cofinal maximal supersets includes much more elements as $\mathcal{QM}$ or the atomic hh-simple sets. E.g. the set from 5.3.4 is such a set , but not hh-simple .

<u>Lemma 8.3.3</u>: Every r.e. T-degree $\bar{d}$ with $\bar{d} \neq 0$ includes simple sets having atomless and also maximal supersets .
<u>Proof</u>: This follows immediately from 8.2.5 , 8.2.6 and $SL_1 \subseteq \mathring{SL}_{1.5}$.

From the above results we get the following summary :

T-degrees of atomless r.e. sets	T-degrees of r.e. sets with cofinal maximal sets	T-degrees of simple sets with atomless and maximal supersets
$\overline{L}_2$	H^1	$\mathcal{Q}_{T,r.e.} \setminus \{0\}$

8.4 Reducibility degrees of the maximal supersets of a recursively enumerable set . Let X be a coinfinite r.e. set . What can said about the reducibility degrees of the r.e. supersets of X ? If X is not simple , then of course all degrees appear among the supersets , since X is contained in a coinfinite recursive set .

The following Theorem gives an answer also for the case that X is a simple set :

Theorem 8.4.1 (Stob)/St,82b/: Let S be a Σ^0_3 set . Suppose

$$(\forall i)(i \in S \longrightarrow W_i \text{ is no infinite and coinfinite recursive set}) .$$

Then there is a r.e. set A , s.t.

$$(\forall i)(i \in S \longrightarrow (W_i \text{ is not recursive} \longrightarrow W_i \not\leq_T A)).$$

In /St,82b/ it is announced that the set A in 8.4.1 can be chosen to be high .

Corollary 8.4.2 (Stob): A coinfinite r.e. set B is simple iff not every high degree has a representant in $\mathcal{E}(B)$.
Proof: The set $\{e : B \subseteq W_e\}$ belongs to Π^0_2 , hence to Σ^0_3 and includes not coinfinite recursive sets , if B is simple . By 8.4.1 there is a high set A with $W_e \not\leq_T A$ for all W_e with $B \subseteq W_e$. Thus in particulary no set from $\mathcal{E}(B)$ has the same T-degree as A .

Since there are maximal sets with the same T-degree as A from 8.4.1 , such maximal sets are not supersets of this B .

Corollary 8.4.3: Let $\leq_r$ be any effective reducibility and B a simple set . Then there are maximal sets , s.t. their r-degrees has no representant in $\mathcal{E}(B)$.

Remark. For the m-reducibility we have the following property for nonsimple r.e. sets : If for a r.e. set X the m-degrees of maximal sets in $\mathcal{E}(X)$ which are co-r.e. (i.e. $M \setminus X$ is co-r.e.) are all m-degrees with maximal sets , then X is already the creative set. This follows immediately from 5.1.3 , 3), since these maximal sets are m-reducibility to X .

9. INDEX SETS AND MAXIMAL SETS

A special branch of the theory of recursively enumerable sets is the estimation of index sets of classes of recursively enumerable sets .

A subset I of ω is called <u>index set</u> if

$$(e \in I \wedge W_e = W_f) \longrightarrow f \in I \quad .$$

If $\mathfrak{X}$ is a class of recursively enumerable sets we write IS($\mathfrak{X}$) for $\{ e : W_e \in \mathfrak{X} \}$.

The central problem of this topic is to characterize the set IS($\mathfrak{X}$) inside the arithmetical hierarchy , i.e. to which of the classes Δ_n^o , Σ_n^o or Π_n^o IS($\mathfrak{X}$) belongs , where n is the minimal index.

This estimations are significant from the algorithm - theoretic view point . If IS($\mathfrak{X}$) belongs e.g. to a high arithmetical class , then we need only weak informations for constructing a set belonging to $\mathfrak{X}$, but for it , it is more difficult to decide if W_e is an element of $\mathfrak{X}$ or not .

The index set estimation can be done for every class of recursively enumerable sets defined anyhow. But also the converse is important . Thus e.g. in point 8 there are already given consequences which follow from some index set properties (see the definition of semilow sets) .

In this point above all the index sets of the class of maximal sets and classes of recursively enumerable sets nearly connected with the maximal sets will be analysed .

9.1 Recursive approximations of arithmetical relations, Σ_n^o - complete and Π_n^o - complete sets

9.2 Index sets of the maximal sets and other classes of recursively enumerable sets

9.3 Pairs of index sets algorithmically uniform separable

9.4 Recursive arrays including all maximal sets

9.1 Recursive approximations of arithmetical relations, Σ_n^0 - complete and Π_n^0 - complete sets. For the theory of index sets a generalization of the creative sets (see REDUCIBILITY THEORY) plays an important role.

Definition 9.1.1 : A set X of natural numbers is called Σ_n^0 - complete for $n \geq 1$ if X belongs to Σ_n^0 and

$$(\forall Y \in \Sigma_n^0)(\exists f \text{ rec. fct})(n \in Y \longleftrightarrow f(n) \in X) \quad .$$

Analogously the notion of Π_n^0 - complete is defined (where in the definition 9.1.1 in all places Σ_n^0 is replaced by Π_n^0)

We see directly from the definition that the Σ_1^0 - complete sets are exactly the creative sets . Further it is well-known and not difficult to prove that for every $n \geq 1$ there are Σ_n^0 - complete sets (and thus by taking the complement of a Σ_n^0 - complete set that Π_n^0 - complete sets exist).

We shall see that the most index sets are Σ_n^0 - complete or Π_n^0 - complete . One obvious property of index sets which are Σ_n^0 - complete is that they do not belong to a smaller arithmetical class and thus Σ_n^0 is the best estimation for this class .

Suppose it is given a class $\mathfrak{X}$ of r.e. sets . The general strategy to calculate IS($\mathfrak{X}$) consists of two parts . In the first part it will be given an upper estimation of IS($\mathfrak{X}$) inside the arithmetical hierarchy , if such an estimation exists and in the second one a lower estimation of IS($\mathfrak{X}$) will be done .

To show that a set IS($\mathfrak{X}$) belongs e.g. to Σ_n^0 we have to show that there is a formula $\varphi(x)$ in the language of the arithmetic ,s.t.

$$\text{IS}(\mathfrak{X}) = \{ e : \mathfrak{N} \models \varphi[e] \}$$

and φ has the form

$$\exists x_1^1 \dots \exists x_{k_1}^1 \forall x_1^2 \dots \forall x_{k_2}^2 \exists x_1^3 \dots Q x_1^n \dots Q x_{k_n}^n \psi(x_1^1, \dots, x_{k_n}^n, e),$$

where Q is $\forall$ if n is even and $\exists$ for n odd and ψ has only bounded quantifiers .

Here we use the fact that " $x \in W_e$ " and " $\varphi_e(x) \simeq y$ " can be defined by formulas of the form

$$\exists x_1 \dots \exists x_n \chi$$

with χ as ψ above .

To show that an index set I from Σ_n^0 is Σ_n^0 - complete it is sufficient for every Σ_n^0 - relation Q to construct a recursiv array $(U_n)_{n \geqslant 0}$, s.t.

$$(\forall x)(x \in Q \longleftrightarrow U_x \in \{ W_e : e \in I \}) .$$

By the S_{mn} - Theorem , see the Introduction , from this the Σ_n^0 - completeness follows .

While the description of an index set in terms of a number theoretic formulas in general is relatively easily , the second part is the complicated one .

For this it is necessary to approximate arithmetical relations by recursive objects . We give here a rough description of such approximations.

Let $R(z, x_1, \dots, x_n)$, $n \geqslant 2$ be a recursive relation . By meaning of R a function λ_z can be defined as follows :

Let $\lambda_z(x_1, \dots, x_{n-2}, s)$ be the number

$$\max \{ z : (\forall z' \leqslant z)(\exists y)(z' + y < s \wedge \\ \wedge R(z, x_1, \dots, x_{n-2}, z', y)) \} .$$

$\max \emptyset =_{df} 0$) .

We see that

- λ_z is a recursive function
- λ_z is (weakly) monotone in s with $\lambda_z(x_1, \dots, x_{n-2}, 0) = 0$

Let $\lambda_z(x_1,\dots,x_{n-2})$ be $\lim_s \lambda_z(x_1,\dots,x_{n-2},s)$. λ_z can be also ω .

$$(\forall x_{n-1})(\exists x_n)R(z,x_1,\dots,x_{n-2},x_{n-1},x_n) \longleftrightarrow \lambda_z(x_1,\dots,x_{n-2}) = \omega \ .$$

From this we get the following approximations :

1° $Q \in \Sigma_3^o$, $Q(i) \longleftrightarrow (\exists x_1)(\forall x_2)(\exists x_3)R(i,x_1,x_2,x_3)$

$$Q(i) \longleftrightarrow (\exists j)(\ \lambda_i(j) = \omega\ ) \ .$$

2° $Q \in \Pi_4^o$, $Q(i) \longleftrightarrow (\forall x_1)(\exists x_2)(\forall x_3)(\exists x_4)\ R(i,x_1,\dots,x_4)$

$$Q(i) \longleftrightarrow (\forall j)(\exists k)(\ \lambda_i(j,k) = \omega\ )$$

3° $Q \in \Sigma_5^o$, $Q(i) \longleftrightarrow (\exists x_1)(\forall x_2)(\exists x_3)(\forall x_4)(\exists x_5)$ $R(i,x_1,\dots,x_5)$

$$Q(i) \longleftrightarrow (\exists j)(\forall k)(\exists l)(\ \lambda_i(j,k,l) = \omega\ ) \ .$$

An other possibility of approximation is to construct recursive arrays instead of λ_z . Thus e.g. equivalent to 1° is to construct a dupple recursive array $(X_{ij})_{ij\geq o}$ with X_{ij} initial parts of ω , s.t.

$$Q(i) \longleftrightarrow (\forall a.a.j)(\ X_{ij} = \omega\ ) \ .$$

(Let X_{ij} be the set $[0,\lambda_i(j))$ with λ_i from 1°) .

In this terms we have , that if Q is from Σ_2^o there is a recursive array $(X_n)_{n\geq o}$ of initial parts of ω , s.t.

$$Q(i) \longleftrightarrow X_i \text{ is finite} \ .$$

Remarks 9.1.2 . 1) In /So,83/ there are given examples of index sets which are Σ^0_n - complete for $n \geq 1$.
(Let A be an Π^0_n - complete set. Then $NE_A = \{e : W_e \cap A \neq \emptyset\}$ is Σ^0_{n+1} - complete).
2) Not every arithmetical index set is Σ^0_n - complete or Π^0_n - complete for some n . E.g. $\{e : W_e$ has exactly one element $\}$ is Δ^0_2 , but not from Σ^0_1 or Π^0_1 . See also /Ro,p.424/.

9.2 Index sets of the maximal sets and other classes of r.e. sets . We will give here index set estimations of the class of maximal sets and classes of r.e. sets connected with this class. Further by means of automorphisms and reducibilities other classes of r.e. sets and thus their index sets can be defined . This also will be described roughly here .

At first we calculate the set $\{e : W_e$ is maximal $\}$. We see that W_e is maximal iff

$$(\forall f)(\ W_e \subseteq W_f \longrightarrow (\ W_e =^* W_f \vee W_f =^* \omega\ )\ ) .$$

Since " $W_e \subseteq W_f$ " is Π^0_2 in e and f , " $W_e =^* W_f$ " as also " $W_e =^* \omega$ " are Σ^0_3 in e and f and in f respectively , the relation " W_e is maximal " is a Π^0_4 relation in e .

Theorem 9.2.1 : The index set $\{e : W_e$ is maximal $\}$ is Π^0_4 - complete .

This result was announced by Yates in /Ya,69/.

The following classes of r.e. sets are nearly connected with the maximal sets :

a) $\{e : W_e$ is q-maximal of order n $\}$, n = 1,2,3,...
b) $\{e : W_e$ is maximal in a recursive set $\}$
c) $\{e : W_e$ is a splitting set of a maximal set $\}$.

It is easy to see that all these three classes have a Σ^0_5 - definition . Are these classes all Σ^0_5 - complete ?

Index sets of orbits of r.e. sets

Let X be a r.e. set . What can said about the index set $\{ e : W_e \cong_{\mathcal{E}} X \}$?

We know that

- the orbit $\{ e : W_e$ is rec. , inf. , coinf. $\}$ is Σ^0_3-complete
- the orbit $\{ e : W_e$ is maximal $\}$ is Π^0_4 - complete .

From 3.4 we know that the classes a) , b) and c) above are also orbits. These would be candidates for Σ^0_5 - complete orbits.

We easily get from Lemma 3.2.2 that the index set of $\sigma_{\mathrm{eff}}(X)$ (i.e. $\{ e : W_e \cong_{\mathrm{eff}} X \}$ for an r.e. set X belongs always to Σ^0_5 and if two automorphic r.e. sets are already automorphic resp. to $\mathrm{Aut}_{\Delta^0_3}(\mathcal{E}^*)$ then $\mathrm{IS}(\sigma_{\mathcal{E}}(X))$ belongs to Σ^0_7 for every X .

Index sets and reducibilities

Let X be an infinite and coinfinite r.e. set. It is known that

$\{ e : W_e \equiv_1 X \}$ is Σ^0_3 - complete /He,85/
$\{ e : W_e \equiv_m X \}$ is Σ^0_3 - complete /Y,66/
$\{ e : W_e \equiv_{tt} X \}$ is Σ^0_3 - complete /Od/,p.73
$\{ e : W_e \equiv_T X \}$ is Σ^X_3 - complete /Y,66/ .

Interesting is the restriction

$$\{ e : W_e - \text{maximal} \wedge W_e \equiv_T M \} ,$$

where M is maximal , in particular when M is T-complete .

Remark. The r.e. sets regarded in point 8 was also analysed under this view point . Jockusch announced that the set $\{ e : W_e \text{ is atomless} \}$ is Π_5^0 - complete .

9.3 Pairs of index sets algorithmically uniform separable .

The index set estimation described in 9.1 and 9.3 can be still improved . By the usual estimation we require only for $i \notin Q$ that U_i does not belong to the family $\mathcal{X}$. This can be still intensified by claiming that for $i \notin Q$ U_i belongs to a family $\mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are disjoint . To this the following def definition :

Definition 9.3.1 : Let X and Y be disjoint subsets of ω . We say that (X,Y) is a (Σ_n^0, Π_n^0)-complete pair ,if $X \in \Sigma_n^0$, $Y \in \Pi_n^0$ and for every (unary) relation $Q \in \Sigma_n^0$ there is a recursive function f , s.t.

$$(9.1) \qquad (\forall x)((x \in Q \longrightarrow f(x) \in X) \wedge (x \notin Q \longrightarrow f(x) \in Y)) .$$

If (X,Y) is a (Σ_n^0, Π_n^0)-complete pair and Z a set from Σ_n^0 with $X \subseteq Z$ and $Z \cap Y = \emptyset$, then Z is Σ_n^0 - complete and $\overline{Z} \in \Pi_n^0$ - complete . Thus in particular X is Σ_n^0 - complete and Y Π_n^0 - complete .

An example of this kind is the pair $(IS(Cof), IS(\varphi))$. It is easy to see that this pair is (Σ_3^0, Π_3^0)-complete . From this we get also Σ_3^0 - completeness of the Index set of recursive sets .

The main Theorem of this subpoint shown by Schwarz is

Theorem 9.3.2 (Schwarz)/Sch/: The pair $(IS(\mathbb{L}), IS(Max))$ is (Σ^0_4, Π^0_4)-complete .

Since maximal sets are not low, these two sets are disjoint. Further it is easy checked that $IS(\mathbb{L})$ has a Σ^0_4-definition. The proof of the completeness is complicated , since this method mentioned in point 3.4 are used .

Remark. Properly speaking in /Sch/ a weaker result was shown, namely that $(IS(\mathbb{L}), IS(\overline{S\mathbb{L}_1}))$ is (Σ^0_4, Π^0_4)-complete. But there is a remark that this can be improved to the pair of the Theorem.

From the Theorem 9.3.2 and the commentary after the definition 9.3.1 we get the following index set estimations :

Corollary 9.3.3: The following index sets :

1) $\{e : W_e \text{ is maximal}\}$
2) $\{e : W_e \text{ is hh-simple}\}$
3) $\{e : W_e \text{ is dense simple}\}$
4) $\{e : W_e \text{ is r-maximal}\}$

are all Π^0_4-complete .

It is not difficult to see that all these four classes have a Π^0_4 - definition. The T-degrees of the first three classes are high by 4.2.2 . The r-maximal sets have also high degrees , see /Le,71/.

The notion of (Σ^0_n, Π^0_n)-complete pair can be weaken. This allows to compare also index sets of different levels (inside the arithmetical hierarchy) .

Let X and Y be disjoint sets . We say that the pair (X, Y) is above all (Σ^0_n, Π^0_n)-pairs if for every Σ^0_n set Q there is a recursive function , s.t. (9.1) is satisfied .

Lemma 9.3.4: The pair $(IS(Rec), IS(Max))$ is above all (Σ^0_3, Π^0_3)-pairs .

This Lemma follows from the observation mentioned in 1.2.2 . Since IS(Rec) has a Σ_3^0 - definition, from Lemma 9.3.4 it follows that this index set is Σ_3^0 - complete .

Another idensification of the usual index set estimation is the following :

Definition 9.3.5: Let X and Y be disjoint subsets of ω . We say that the pair (X,Y) is a Σ_n^0 - complete pair if both sets are from Σ_n^0 and for every pair (Q_1, Q_2) of disjoint Σ_n^0-pairs there is a recursive function f with

$$(\forall x)(\ (x \in Q_1 \longleftrightarrow f(x) \in X)\ \wedge\ (x \in Q_2 \longleftrightarrow f(x) \in Y)\ ) .$$

Analogeously we can define a Π_n^0 - complete pair . For Σ_1^0 - complete pairs we say also creative pair .

We see that also in this case it follows that X is Σ_n^0-complete if X belongs to a Σ_n^0-complete pair . Another property of a Σ_n^0 - complete pair is that such sets are not separable by a Δ_n^0 - set . See for this topic e.g. /Sm/ .

9.4 Recursive arrays including all maximal sets . We investigate in this subpoint a topic which is nearly connected with the index set estimation .

Let given a family of r.e. sets $\mathcal{X}$. Suppose $(U_n)_{n\geq 0}$ is a recursive array which includes all elements of $\mathcal{X}$ as members. Which further r.e. sets , not in $\mathcal{X}$, this array must include yet ?

Let $\mathcal{X}_0$ and $\mathcal{X}_1$ be two classes of r.e. sets with $\mathcal{X}_0 \subseteq \mathcal{X}_1$ and $(U_n)_{n\geq 0}$ a recursive array . We write $\mathcal{X}_0 \subseteq (U_n)_{n\geq 0} \subseteq \mathcal{X}_1$ if

$$(\forall X \in \mathcal{X}_0)(\exists n)(\ X = U_n\ )\ \wedge\ (\forall n)(\exists Y)(\ Y \in \mathcal{X}_1 \wedge U_n = Y).$$

Theorem 9.4.1(Yates): For every class $\mathfrak{X}$ holds :
There is a recursive array $(U_n)_{n\geq 0}$ with

$$\mathfrak{X} \subseteq (U_n)_{n\geq 0} \subseteq \mathfrak{X} \cup \text{Fin}$$

iff IS($\mathfrak{X}$) belongs to Σ_3^0 .

Thus by 9.2.1 there is no recursive array $(U_n)_{n\geq 0}$ with Max $\subseteq (U_n)_{n\geq 0} \subseteq$ Max $\cup$ Fin . But we can give a more precise evaluation .

Theorem 9.4.2: Every recursive array which includes the class Max must have nonsimple sets and also simple , not maximal sets.

10. COHESIVE SETS

In this point the notion of cohesive set defined already in point 1.1 will be regarded in general . Since " the most " cohesive sets are not co-r.e. as we shall see, this consideration is not already included in the analyse of the maximal sets.

10.1 The structure of the cohesive sets . The existence of such special subsets of ω follows from the existence of maximal sets . But if we do not require that a set except to be cohesive still has to be co-r.e. , then the proof of the existence can be provided much more easier even with additionally properties.

Theorem 10.1.1 (Dekker,Myhill)/Ro/,p.296: Every infinite subset of ω contains a cohesive subset .

$\approx_c$ - equivalence between cohesive sets

Let X_1 and X_2 be cohesive sets . We write $X_1 \approx_c X_2$ if

$$(\forall e)(X_1 \subset^* W_e \longrightarrow X_2 \subset^* W_e) \quad .$$

It is easy to see that $\approx_c$ is an equivalence relation in the class of cohesive sets .

Lemma 10.1.2: 1) Every equivalence class (of cohesive sets) resp. to $\approx_c$ united with Fin forms an ideal in $\mathcal{P}(\omega)$.
2) For cohesive sets X_1 and X_2 we have $X_1 \approx_c X_2$ or $X_1 \cap X_2 =^* \emptyset$.

3) An equivalence class resp. to $\approx_c$ has a maximal element (mod Fin) iff the maximal element is co-r.e. .
Hence , iff the complement of this ~~maximal~~ cohesive element is a maximal set .
4) There are Continuum many equivalence classes resp. to $\approx_c$.

Cohesive subsets of a set

Let X be a subset of ω . How many (pairwise) not $\approx_c$ - equivalent cohesive sets are included in X ? Let c(X) be

$$\sup \{ \mathrm{card}(\mathcal{X}) : \mathcal{X} \subseteq \mathcal{P}(\omega) \wedge (\forall X \in \mathcal{X})(X \text{ is cohesive}) \wedge \wedge (\forall X_1 \in \mathcal{X})(\forall X_2 \in \mathcal{X})(X_1 \neq X_2 \rightarrow X_1 \not\approx_c X_2) \} \tag{10.1}$$

Lemma 10.1.3: For every $X \in \mathcal{P}(\omega)$ holds $c(X) \leq \omega$ or $c(X) = 2^{\aleph_0}$.

Observe that 10.1.3 holds without the Continuum Hypothesis .

Further for every α with $0 \leq \alpha \leq \omega$ or $\alpha = 2^{\aleph_0}$ there is a set X with $c(X) = \alpha$.

Respectively to the cardinality (10.1) some subsets of α can be characterized

X is finite $\longleftrightarrow$ $c(X) = 0$
X is cohesive $\longleftrightarrow$ $c(X) = 1$
X is q-cohesive $\longleftrightarrow$ $c(X) < \omega$
X includes an infinite r.e. set $\longrightarrow$ $c(X) = 2^{\aleph_0}$

X is called general - cohesive (g-cohesive) if X is not q-maximal cohesive and .

$$(\forall e)((W_e \cap X \text{ is finite or q-cohesive}) \vee (\overline{W_e} \cap X \text{ is finite or q-cohesive})) .$$

X is g-cohesive $\longrightarrow$ $c(X) = \omega$ (But not converse) .

Let X and Y be subsets of ω . We write $X \leq_k Y$ if

$$(\forall C \text{ cohesive})(\ C \subseteq X \longrightarrow (\exists C' \text{ cohesive})(\ C \approx_c C' \wedge \\ \wedge\ C' \subseteq Y))$$

and $X \approx_k Y$ if $X \leq_k Y \wedge Y \leq_k X$.

<u>Lemma 10.1.4:</u> For every infinite set X there is an subset Y of X which is immune and $X \approx_k Y$.

<u>q - cohesive sets</u>

By using of the q-cohesive sets we get a class of (finite) distributive lattices which includes the class of finite Boolean algebras .

A distributive lattice $\mathscr{L}$ is called <u>separated</u> /La,71/ if in $\mathscr{L}$ the Reduction principle holds .

In /La,70/ is shown that a finite distributive lattice $\mathscr{L}$ is separated iff there is a subfamily $\mathcal{N}$ of $\mathscr{L}$ with

1° $\mathcal{N}$ is a tree under the inclusion (i.e. for $x, y_1, y_2 \in \mathcal{N}$ holds $(\ x \leq y_1 \wedge x \leq y_2) \longrightarrow (\ y_1 \leq y_2 \wedge y_2 \leq y_1)\ )$.

2° Every element of $\mathscr{L}$ is a finite union of elements of $\mathcal{N}$.

<u>Theorem 10.1.5:</u> The class of lattices $(\ \mathcal{E} \mid X)^*$ where X is q-cohesive coincides with the class of all finite , separated distributive lattices .

<u>Remark.</u> The Theorem remains true , if we take only q-cohesive sets which are Δ^0_2. If the q-cohesive set X is co-r.e. , then by 2.2.1 $(\ \mathcal{E} \mid X)^*$ is a finite Boolean algebra .

In 2.3 it was shown that

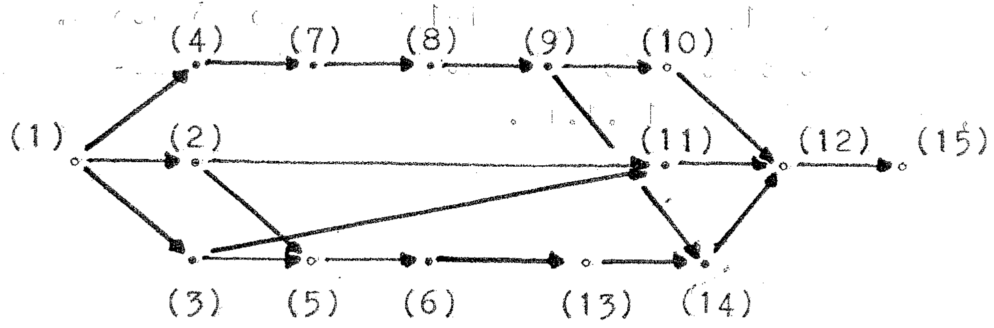
$$Q\mathcal{M} \cap \text{Co-Mon} = Q\mathcal{M} \cap \text{Co-1-1} = \underline{\text{Max}} \quad .$$

For q-cohesive sets this is not so . There are q-cohesive sets of order n ($n \geq 1$) which are 1-1 and also monotone. This follows from the fact that major subsets of maximal sets are co-monotone /Mad,Rob/ and from 10.1.1 .

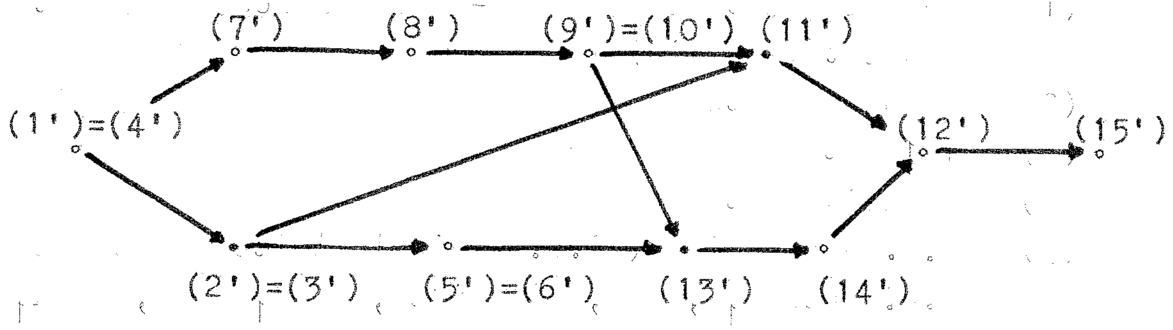
10.2 Classification of the immune sets . It will be here still given a classification of the immune sets. This subdivision of the class of immune sets is a combination of that in /Ro/,p.312 and of that in /Mad,Rob/.

The following classification which contain 14 proper subclasses of the class of immune sets is complete.

(1) Cohesive sets in an $\approx_c$ - equivalence class with a maximal element (mod Fin)
(2) Monotone sets
(3) 1-1 sets
(4) Cohesive sets
(5) r.e. - cohesive (i.e. X is infinite and there are no r.e. sets X_1 , X_2 , disjoint , $X \subseteq X_1 \cup X_2$, $X \cap X_1$ and $X \cap X_2$ are both infinite)
(6) r-cohesive
(7) q-cohesive
(8) q-cohesive or g-cohesive
(9) strongly hh-immune (i.e. there is no recursive array $(S_n)_{n \geq 0}$ disjoint and $S_i \cap X \neq \emptyset$ for all $i \geq 0$)
(10) hh-immune
(11) dense immune
(12) h-immune
(13) strongly h-immune
(14) finitely strongly h-immune
(15) immune



Observe that the class of dense immune sets (11) even does not include the class of cohesive sets (4). But if we restrict all above classes the subclasses having only the co-r.e. sets of these classes , then we get the following classification :



where (n') denotes the subclass of (n) (n=1,...,15) with the co-r.e. sets of (n) .

NOTATIONS

Introd. 1

Introd. 2

Introd. 3

Introd. 4

6. point

7. point

8. point

9. point

10. point

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